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On some applications of relative (p, q)-th order for rating the growths of composite entire functions

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Abstract. The main aim of this paper is to study some comparative growth properties of composite entire functions on the basis of relative (p, q)-th order and relative (p, q)-th lower order of entire function with respect to another entire function where p and q are any two positive integers.

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1 Introduction, Definitions and Notations

We assume that the reader is familiar with the fundamental results and the standard notations of the theory of entire functions which are available in [22]. Let $f(z)$ be an entire function defined in the open complex plane $\mathbb{C}$ and $\mathbb{N}$ be the sets of all positive integers. The maximum modulus function $M_f(r)$ is defined as $M_f(r) = \max_{|z|=r}|f(z)|$. Since $M_f(r)$ is strictly increasing and continuous, its inverse function exists. For another entire function $g(z)$, $M_g(r)$ is defined and the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to +\infty$ is called the growth of $f(z)$ with respect to $g(z)$ in terms of their maximum moduli. The maximum term $\mu_f(r)$ of $f(z)$ can be defined as $\mu_f(r) = \max_{n\geq 0}(|a_n|r^n)$. In fact $\mu_f(r)$ is much weaker than $M_f(r)$ in some sense. So from another angle of view $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to +\infty$ is also called the growth of $f(z)$ with respect to $g(z)$ where $\mu_g(r)$ denotes the maximum term of entire function $g(z)$.

If $f(z)$ and $g(z)$ are entire functions, then the iteration of $f(z)$ with respect to $g(z)$ is defined as follows (see [14]):

$$\begin{aligned}
f_1(z) &: = f(z);\\
f_2(z) &: = f(g(z)) = f(g_1(z));\\
f_3(z) &: = f(g(f(z))) = f(g(f_1(z))) = f(g_2(z));\\
&\quad \cdots\\
f_\eta(z) &: = f(g(f\cdots(h(z))\cdots)) \quad (\eta \in \mathbb{N}),
\end{aligned}$$

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where $h(z) = f(z)$ when η is odd and $h(z) = g(z)$ when η is even.

Similarly one defines

$$\begin{aligned} g_1(z) &: = g(z); \\ g_2(z) &: = g(f(z)) = g(f_1(z)); \\ &\quad \cdots \\ g_\eta(z) &: = g(f(g_{\eta-2}(z))) = g(f_{\eta-1}(z)) \quad (\eta \in \mathbb{N}). \end{aligned}$$

It is obvious that $f_\eta(z)$ and $g_\eta(z)$ ($\eta \in \mathbb{N}$) are all entire functions. Similarly for another two entire functions $l(z)$ and $k(z)$, one can easily define $l_\xi(z)$ and $k_\xi(z)$ where $\xi \in \mathbb{N}$. Further we assume that throughout the present paper $\eta, \xi \in \mathbb{N}$ always denote the even numbers.

For $x \in [0,\infty)$ and $k \in \mathbb{N}$, define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$ with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Now considering this, let us recall that Juneja et al.[13] defined the (p,q)-th order and (p,q)-th lower order of an entire function respectively, as follows:

Definition 1. [13] The (p,q)-th order and (p,q)-th lower order of an entire function $f(z)$ are defined as:

$$\rho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p and q always denote positive integers with $p \geq q$.

Extending the notion of (p,q)-th order, Shen et al.[20] introduced the new concept of $[p,q]$-φ order of an entire function where $p \geq q$. Later on, combining the definition of (p,q)-th order and $[p,q]$-φ order, Biswas (see, e.g.,[3]) redefined the (p,q)-th order of an entire function without restriction $p \geq q$.

In this connection we just recall the following definition where we will give a minor modification to the original definition (see e.g.[13]):

Definition 2. An entire function $f(z)$ is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and ${}^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ otherwise. Moreover if $0 < \rho^{(p,q)}(f) < +\infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = +\infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases} .$$

Similarly for $0 < \lambda^{(p,q)}(f) < +\infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = +\infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases} .$$

An entire function $f(z)$ of index-pair (p, q) is said to be of regular (p, q) growth if its (p, q)-th order coincides with its (p, q)-th lower order, otherwise $f(z)$ is said to be of irregular (p, q) growth.

However the above definition is very useful for measuring the growth of entire functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire function f. For details about generalized order one may see [17]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire function $f(z)$.

Since for $0 \leq r < R$,

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R) \ \{\text{cf.}[19]\},$$

it is easy to see that

$$\rho_f(p,q) = \limsup_{r\to+\infty}\frac{\log^{[p]}\mu_f(r)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to+\infty}\frac{\log^{[p]}\mu_f(r)}{\log^{[q]} r},$$

where $p, q \in \mathbb{N}$.

For entire functions, the notions of their growth indicators such as order are classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of relative orders of entire functions introduced by Bernal [1, 2] as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions in the light of their relative orders are the prime concern of this paper. In fact some light has already been thrown on such type of works (see [3] to [7] and [9] to [11]). Extending the notion of relative order of entire function as introduced Bernal [1,2], Lahiri and Banerjee [15] introduced the definition of relative (p, q)-th order of entire functions as follows.

Definition 3. [15] Let p and q be any two positive integers with $p \geq q$. The relative (p, q)-th order of $f(z)$ with respect to $g(z)$ is defined by

$$\rho_g^{(p,q)}(f) = \limsup_{r\to+\infty}\frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

Then $\rho_{\exp z}^{(p,q)}(f) = \rho_f(p,q)$ and $\rho_g^{t(k+1,1)}(f) = \rho_g^{[k]}(f)$ for any $k \geq 1$.

Sánchez Ruiz et al.[16] gave a more natural definition of relative (p, q)-th order of an entire function in the light of index-pair. In the next definition, we will give a minor modification to the original definition (see e.g.[16]):

Definition 4. Let $f(z)$ and $g(z)$ be any two entire functions with index-pairs (m,q) and (m,p) respectively where p,q and m are all positive integers. Then the relative (p,q)-th order of $f(z)$ with respect to $g(z)$ is defined as

$$\rho_g^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

Similarly one can define the relative (p,q)-th lower order of an entire function $f(z)$ with respect to another entire function $g(z)$ denoted by $\lambda_g^{(p,q)}(f)$ where p and q are any two positive integers in the following way:

$$\lambda_g^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

In fact Definition 4 improves Definition 3 ignoring the restriction $p \geq q$.

If $f(z)$ and $g(z)$ have got index-pair $(m,1)$ and (m,k), respectively, then Definition 4 reduces to generalized relative order of $f(z)$ with respect to $g(z)$. If the entire functions $f(z)$ and $g(z)$ have the same index-pair $(p,1)$ where p is any positive integer, we get the definition of relative order introduced by Bernal [1,2] and if $g(z) = \exp^{[m-1]} z$, then $\rho_g(f) = \rho_f^{[m]}$ and $\rho_g^{(p,q)}(f) = \rho_f(m,q)$. Further if $f(z)$ is an entire function with index-pair $(2,1)$ and $g(z) = \exp z$, then Definition 4 becomes the classical one given in [21].

An entire function $f(z)$ for which relative (p,q)-th order and relative (p,q)-th lower order with respect to another entire function $g(z)$ are the same is called a function of regular relative (p,q) growth with respect to $g(z)$. Otherwise, $f(z)$ is said to be irregular relative (p,q) growth with respect to $g(z)$.

In terms of maximum terms of entire functions, Definition 4 can be reformulated as:

Definition 5. For any positive integer p and q, the growth indicators $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ of an entire function $f(z)$ with respect to another entire function $g(z)$ are defined as:

$$\rho_g^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} \mu_g^{-1}(\mu_f(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} \mu_g^{-1}(\mu_f(r))}{\log^{[q]} r}.$$

In fact, the equivalence of Definition 4 and Definition 5 has been established in [4].

In this paper we establish some newly developed results related to the growth rates of iteration of entire functions on the basis of relative (p,q)-th order and relative (p,q)-th lower order improving some earlier results where p and q are any two positive integers.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[8] Let $f(z)$ and $g(z)$ be any two entire functions with $g(0)=0$. Let β satisfy $0<\beta<1$ and $c(\beta)=\frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*

$$M_f(c(\beta)M_g(\beta r)) \leq M_{f\circ g}(r) \leq M_f(M_g(r)).$$

In addition if $\beta=\frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\Big(\frac{1}{8}M_g\Big(\frac{r}{2}\Big)\Big).$$

Lemma 2. *[18] Let $f(z)$ and $g(z)$ be any two entire functions. Then for every $\alpha>1$ and $0<r<R$,*

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha-1}\mu_f\Big(\frac{\alpha R}{R-r}\mu_g(R)\Big).$$

Lemma 3. *[18] If $f(z)$ and $g(z)$ are any two entire functions with $g(0)=0$, then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\Big(\frac{1}{8}\mu_g\Big(\frac{r}{4}\Big)\Big).$$

Lemma 4. *[6] Suppose $f(z)$ is an entire function and $\alpha>1$, $0<\beta<\alpha$. Then for all sufficiently large r,*

$$M_f(\alpha r) \geq \beta M_f(r).$$

Lemma 5. *[12] If $f(z)$ is an entire function and $\alpha>1$, $0<\beta<\alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta\mu_f(r).$$

3 Main Results

In this section we present the main results of the paper.

Theorem 1. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l,q) and (l,p) respectively such that $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<+\infty$ and $g(z)$ be an entire function with non-zero (m,n)-th order where l,p,q,m and n are all positive integers. Then for every positive constant A,*

$$(i)\quad \limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]}r))}{\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]}r^A))} = +\infty \text{ if } q=m \text{ and } n=l,$$

$$(ii)\quad \limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]}r))}{\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]}r^A))} = +\infty \text{ if } q>m \text{ and } q+n-m\leq l$$

and

$$(iii)\quad \limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))}{\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]}r^A))} = +\infty \text{ if } q\leq m-1,\ n\leq l \text{ and } 0<A<\rho^{(m,n)}(g).$$

Proof. From the definition of $\rho_h^{(p,q)}(f)$, in terms of maximum terms, we obtain for arbitrary positive $\varepsilon(>0)$ and for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]} r^A)) \leq (\rho_h^{(p,q)}(f)+\varepsilon)r^A. \tag{1}$$

Also from the definition of (m,n)-th order of $g(z)$ in terms of maximum terms, we get for a sequence of values of r tending to infinity that

$$\log^{[m]}\mu_g\Big(\frac{\exp^{[q+n+1-m]} r}{(196)^{\frac{n}{2}}}\Big) \geqslant (\rho^{(m,n)}(g)-\varepsilon)\log^{[n]}\Big(\frac{\exp^{[q+n+1-m]} r}{(196)^{\frac{n}{2}}}\Big)$$

$$i.e.,\ \log^{[m]}\mu_g\Big(\frac{\exp^{[q+n+1-m]} r}{(196)^{\frac{n}{2}}}\Big) \geqslant (\rho^{(m,n)}(g)-\varepsilon)\exp^{[q+1-m]} r + O(1)$$

$$i.e.,\ \log^{[q-m]}\log^{[m]}\mu_g\Big(\frac{\exp^{[q+n+1-m]} r}{(196)^{\frac{n}{2}}}\Big) \geqslant \log^{[q-m]}((\rho^{(m,n)}(g)-\varepsilon)\exp^{[q+1-m]} r + O(1))$$

$$i.e.,\ \log^{[q]}\mu_g\Big(\frac{\exp^{[q+n+1-m]} r}{(196)^{\frac{n}{2}}}\Big) \geqslant \exp r + O(1), \tag{2}$$

and

$$\log^{[m]}\mu_g\Big(\frac{\exp^{[n-1]} r}{(196)^{\frac{n}{2}}}\Big) \geqslant (\rho^{(m,n)}(g)-\varepsilon)\log^{[n]}\Big(\frac{\exp^{[n-1]} r}{(196)^{\frac{n}{2}}}\Big)$$

$$i.e.,\ \log^{[m]}\mu_g\Big(\frac{\exp^{[n-1]} r}{(196)^{\frac{n}{2}}}\Big) \geqslant (\rho^{(m,n)}(g)-\varepsilon)\log r + O(1)$$

$$i.e.,\ \log^{[m-1]}\mu_g\Big(\frac{\exp^{[n-1]} r}{(196)^{\frac{n}{2}}}\Big) \geqslant r^{(\rho^{(m,n)}(g)-\varepsilon)} + O(1). \tag{3}$$

Case I. Let $q=m$ and $n=l$.

Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 3 and Lemma 5 for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{n}{2}}r)) \geq \log^{[p]}\mu_h^{-1}(\mu_f(\mu_{g_{\eta-1}}(r))) \tag{4}$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{n}{2}}r)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)\log^{[q]}\mu_{g_{\eta-1}}(r)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)\log^{[m]}\mu_g(\mu_{f_{\eta-2}}(r)) \tag{5}$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)\log^{[n]}\mu_{f_{\eta-2}}(r), \tag{6}$$

Applying (6) to continue this process, we have

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)(\lambda^{(l,q)}(f)-\varepsilon)\log^{[q]}\mu_{g_{\eta-3}}(r),$$

and so on.

We finally have the following inequality for all sufficiently large values of r,

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq$$
$$(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}\log^{[q]}\mu_g(r). \tag{7}$$

Now from above we get for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]}r)) \geq$$
$$(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}\log^{[q]}\mu_g\Big(\frac{\exp^{[n+1]}r}{(196)^{\frac{\eta}{2}}}\Big).$$

Now it follows from above for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]}r)) \geq$$
$$(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(m,n)}(g)-\varepsilon)\log^{[n]}\Big(\frac{\exp^{[n+1]}r}{(196)^{\frac{\eta}{2}}}\Big)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]}r)) \geq$$
$$(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(m,n)}(g)-\varepsilon)\exp r+O(1). \tag{8}$$

Case II. Let $q > m$ and $q+n-m \leq l$.

In view of (5), we obtain for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)\log^{[q-m]}\log^{[m]}\mu_g(\mu_{f_{\eta-2}}(r))$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)\log^{[q-m]}((\lambda^{(m,n)}(g)-\varepsilon)\log^{[n]}(\mu_{f_{\eta-2}}(r)))$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)\log^{[q+n-m]}\mu_{f_{\eta-2}}(r)+O(1)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)\log^{[l]}\mu_{f_{\eta-2}}(r)+O(1), \tag{9}$$

Applying (9) to continue this process, we have

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(l,q)}(f)-\varepsilon)\log^{[q]}\mu_{g_{\eta-3}}(r)+O(1),$$

and so on.

We finally have the following inequality,

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}((196)^{\frac{\eta}{2}}r)) \geq$$
$$(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}\log^{[q]}\mu_g(r)+O(1). \tag{10}$$

Now from (2) and (10), it follows for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]}r)) >$$

$$(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}\log^{[q]}\mu_g\Big(\frac{\exp^{[q+n+1-m]}r}{(196)^{\frac{\eta}{2}}}\Big)+O(1).$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]}r))\geq(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}\exp r+O(1). \tag{11}$$

Case III. Let $q\leq m-1$, $n\leq l$ and $0<A<\rho^{(m,n)}(g)$.

In view of (3) and (7), we obtain for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))>$$
$$(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}\log^{[q]}\mu_g\Big(\frac{\exp^{[n-1]}r}{(196)^{\frac{\eta}{2}}}\Big)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))>$$
$$(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}\log^{[m-1]}\mu_g\Big(\frac{\exp^{[n-1]}r}{(196)^{\frac{\eta}{2}}}\Big)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))>$$
$$(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}r^{(\rho^{(m,n)}(g)-\varepsilon)}+O(1). \tag{12}$$

Now combining (1) and (8) of Case I it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]}r))}{\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]}r^A))}\geq$$
$$\frac{(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(m,n)}(g)-\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(m,n)}(g)-\varepsilon)\exp r+O(1)}{(\rho_h^{(p,q)}(f)+\varepsilon)r^A}.$$

Since $\frac{\exp r}{r^A}\to+\infty$ as $r\to+\infty$, then from above it follows that

$$\limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]}r))}{\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]}r^A))}=+\infty,$$

from which the first part of the theorem follows.

Again combining (1) and (11) of Case II we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]}r))}{\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]}r^A))}\geq\frac{(\lambda_h^{(p,q)}(f)-\varepsilon)(\lambda^{(l,q)}(f)-\varepsilon)^{\frac{\eta}{2}-1}\exp r+O(1)}{(\rho_h^{(p,q)}(f)+\varepsilon)r^A}$$

$$i.e.\ \limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]}r))}{\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]}r^A))}=+\infty.$$

This establishes the second part of the theorem.

Once more, it follows from (1) and (12) of Case III for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))} >$$

$$\frac{(\lambda_h^{(p,q)}(f) - \varepsilon)(\lambda^{(m,n)}(g) - \varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(l,q)}(f) - \varepsilon)^{\frac{\eta}{2}-1} r^{(\rho^{(m,n)}(g)-\varepsilon)} + O(1)}{(\rho_h^{(p,q)}(f) + \varepsilon) r^A}. \tag{13}$$

As $A < \rho^{(m,n)}(g)$ we can choose $\varepsilon\,(> 0)$ in such a way that

$$A < \rho^{(m,n)}(g) - \varepsilon. \tag{14}$$

Thus from (13) and (14) we get that

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))} = +\infty.$$

This proves the third part of the theorem.

Thus the theorem follows. □

In view of Theorem 1 the following theorem can be carried out:

Theorem 2. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l, q) and (l, p) respectively such that $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < +\infty$ and $g(z)$ be an entire function with positive (m, n)-th lower order where l, p, q, m and n are all positive integers. Then for every positive constant A,*

$$(i) \quad \lim_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]} r))}{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))} = +\infty \text{ if } q = m \text{ and } n = l,$$

$$(ii) \quad \lim_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]} r))}{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))} = +\infty \text{ if } q > m \text{ and } q + n - m \le l$$

and

$$(iii) \quad \lim_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))} = +\infty \text{ if } q \le m-1,\ n \le l \text{ and } 0 < A < \rho^{(m,n)}(g).$$

Theorem 3. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l, q) and (l, p) respectively such that $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < +\infty$. Suppose $g(z)$ is an entire function with positive (m, n)-th order $\rho^{(m,n)}(g)$ and finite relative (p, n)-th order $\rho_k^{(p,n)}(g)$ with respect to another entire function $k(z)$ where l, p, q, m and n are all positive integers. Then for every positive constant A,*

$$(i) \quad \limsup_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]} r))}{{}^{[p]}\mu_k^{-1}(\mu_g(\exp^{[n]} r^A))} = +\infty \text{ if } q = m \text{ and } n = l,$$

$$(ii)\ \limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]}r))}{{}^{[p]}\mu_k^{-1}(\mu_g(\exp^{[n]}r^A))} = +\infty \text{ if } q>m \text{ and } q+n-m\leq l$$

and

$$(iii)\ \limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))}{{}^{[p]}\mu_k^{-1}(\mu_g(\exp^{[n]}r^A))} = +\infty \text{ if } q\leq m-1, n\leq l \text{ and } 0<A<\rho^{(m,n)}(g).$$

Proof. Suppose $0<A<A_0$.

Case I. Let $q=m$ and $n=l$. Then in view of the first part of Theorem 1, we get for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]}r)) > (\lambda_h^{(p,q)}(f)-\varepsilon)r^{A_0}. \tag{15}$$

Case II. Also let $q>m$ and $q+n-m\leq l$. Then we obtain from the second part of Theorem 1 for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]}r)) > (\lambda_h^{(p,q)}(f)-\varepsilon)r^{A_0}. \tag{16}$$

Case III. Again let $q\leq m-1$, $n\leq l$ and $0<A<\rho^{(m,n)}(g)$. Then we get from the third part of Theorem 1 for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r)) > (\lambda_h^{(p,q)}(f)-\varepsilon)r^{A_0}. \tag{17}$$

Now from the definition of $\rho_h^{(p,n)}(g)$ in terms of maximum terms, we obtain for all sufficiently large values of r that

$$\log^{[p]}\mu_k^{-1}(\mu_g(\exp^{[n]}r^A)) \leq (\rho_k^{(p,n)}(g)+\varepsilon)r^A. \tag{18}$$

Now combining (15) of Case I and (18) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]}r))}{{}^{[p]}\mu_k^{-1}(\mu_g(\exp^{[n]}r^A))} > \frac{(\lambda_h^{(p,q)}(f)-\varepsilon)r^{A_0}}{(\rho_k^{(p,n)}(g)+\varepsilon)r^A}. \tag{19}$$

Since $A_0>A$, from (19) it follows that

$$\limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]}r))}{{}^{[p]}\mu_k^{-1}(\mu_g(\exp^{[n]}r^A))} = +\infty,$$

from which the first part of the theorem follows.

Similarly for $A_0>A$, we obtain from (16) of Case II and (18) for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]}r))}{{}^{[p]}\mu_k^{-1}(\mu_g(\exp^{[n]}r^A))} > \frac{(\lambda_h^{(p,q)}(f)-\varepsilon)r^{A_0}}{(\rho_k^{(p,n)}(g)+\varepsilon)r^A}$$

$$i.e. \limsup_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]} r))}{{}^{[p]}\mu_k^{-1}(\mu_g(\exp^{[n]} r^A))} = +\infty.$$

This establishes the second part of the theorem.

Again it follows from (17) of Case III and (18) for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{{}^{[p]}\mu_k^{-1}(\mu_g(\exp^{[n]} r^A))} \geq \frac{(\lambda_h^{(p,q)}(f)-\varepsilon)r^{A_0}}{(\rho_k^{(p,n)}(g)+\varepsilon)r^A}. \tag{20}$$

Now suppose A_0 is such that $0 < A < A_0 < \rho^{(m,n)}(g)$.

Therefore from (20) we get that

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{\log^{[p]} \mu_k^{-1}(\mu_g(\exp^{[n]} r^A))} = +\infty.$$

This proves the third part of the theorem.

Thus the theorem is established. □

Theorem 4. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l,q) and (l,p) respectively such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < +\infty$. Suppose $g(z)$ is an entire function with positive (m,n)-th lower order $\lambda^{(m,n)}(g)$ and finite relative (p,n)-th order $\rho_k^{(p,n)}(g)$ with respect to another entire function $k(z)$ where l,p,q,m and n are all positive integers. Then for every positive constant A,*

$$(i) \quad \lim_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n+1]} r))}{\log^{[p]} \mu_k^{-1}(\mu_g(\exp^{[n]} r^A))} = +\infty \text{ if } q=m \text{ and } n=l,$$

$$(ii) \quad \lim_{r\to+\infty} \frac{\log^{[p]} \mu_k^{-1}(\mu_{f_\eta}(\exp^{[q+n+1-m]} r))}{\log^{[p]} \mu_h^{-1}(\mu_g(\exp^{[n]} r^A))} = +\infty \text{ if } q>m \text{ and } q+n-m \leq l$$

and

$$(iii) \quad \lim_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{\log^{[p]} \mu_k^{-1}(\mu_g(\exp^{[n]} r^A))} = +\infty \text{ if } q \leq m-1, n \leq l \text{ and } 0 < A < \rho^{(m,n)}(g).$$

The proof of Theorem 4 is omitted as it can be carried out in the line of Theorem 3 and with the help of Theorem 2.

Theorem 5. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l,q) and (l,p) respectively such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < +\infty$ and $g(z)$ be an entire function with finite (m,n)-th order and (m,n)-th lower order where l,p,q,m and n are all positive integers. Then*

$$(i) \quad \limsup_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n]} r))} = +\infty \text{ if } q \geq m, n=l \text{ and } A>1,$$

$$(ii)\ \limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty \text{ if } q=m \text{ or } q\geq m(\neq 1)-1,\ n \geq l \text{ and } A>\lambda^{(m,n)}(g)$$

and

$$(iii)\ \limsup_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p+(m-q)(\frac{\eta}{2}-1)-1]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty \text{ if } m>q+1,\ n\geq l \text{ and } A > \lambda^{(m,n)}(g).$$

Proof. From the definition of $\lambda_h^{(p,q)}(f)$ in terms of maximum terms, we obtain for arbitrary positive $\varepsilon(>0)$ and for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q]} r^A)) \geq (\lambda_h^{(p,q)}(f)-\varepsilon)r^A. \tag{21}$$

Also from the definition of (m,n)-th lower order of $g(z)$ in terms of maximum terms and for $\alpha>1$, $0<\beta<\alpha$, we get for a sequence of values of r tending to infinity that

$$\log^{[m]}\mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]} r\Big) \leq (\lambda^{(m,n)}(g)+\varepsilon)\log^{[n]}\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]} r\Big)$$

$$i.e.,\ \log^{[m]}\mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]} r\Big) \leq \Big(\lambda^{(m,n)}(g)+\varepsilon\Big)\log r+O(1)$$

$$i.e.,\ \log^{[m]}\mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]} r\Big) \leq \log r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)}+O(1) \tag{22}$$

$$i.e.,\ \log^{[m-1]}\mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]} r\Big) \leq r^{(\lambda^{(m,n)}(g)+\varepsilon)}+O(1). \tag{23}$$

Case I. Let $q\geq m$ and $n=l$.

Since $\mu_h^{-1}(r)$ is an increasing function of r, taking $R=\beta r$ in Lemma 2 and in view of Lemma 5 it follows for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}}r\Big)\Big) \leq \log^{[p]}\mu_h^{-1}(\mu_f(\mu_{g_{\eta-1}}(r)))$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}}r\Big)\Big) \leq (\rho_h^{(p,q)}(f)+\varepsilon)\log^{[q]}\mu_{g_{\eta-1}}(r) \tag{24}$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}}r\Big)\Big) \leq (\rho_h^{(p,q)}(f)+\varepsilon)\log^{[m]}\mu_g(\mu_{f_{\eta-2}}(r))$$

$$i.e., \ \log^{[p]} \mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}} r\Big)\Big) \leq$$

$$(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)\log^{[n]} \mu_{f_{\eta-2}}(r), \tag{25}$$

Applying (25) to continue this process, we have

$$\log^{[p]} \mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}} r\Big)\Big) \leq$$

$$(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)(\rho^{(l,q)}(f)+\varepsilon)\log^{[q]} \mu_{g_{\eta-3}}(r),$$

and so on.

We finally have the following inequality for all sufficiently large values of r,

$$\log^{[p]} \mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}} r\Big)\Big) \leq$$

$$(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f)+\varepsilon)^{\frac{\eta}{2}-1}\log^{[q]} \mu_g(r). \tag{26}$$

Now from above we get for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n]} r)) \leq$$

$$(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f)+\varepsilon)^{\frac{\eta}{2}-1}\log^{[q]} \mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}} \exp^{[n]} r\Big) \tag{27}$$

$$i.e., \ \log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n]} r)) \leq$$

$$(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f)+\varepsilon)^{\frac{\eta}{2}-1}\log^{[m]} \mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}} \exp^{[n]} r\Big).$$

Now it follows from above for a sequence of values of r tending to infinity that

$$\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n]} r)) \leq$$

$$(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f)+\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(m,n)}(g)+\varepsilon)r+O(1). \tag{28}$$

Further from (23) and (26), we obtain for a sequence of values of r tending to infinity that

$$\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) <$$

$$(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f)+\varepsilon)^{\frac{\eta}{2}-1}\log^{[m-1]} \mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}} \exp^{[n-1]} r\Big)$$

$$i.e. \ \log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) <$$

$$(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f)+\varepsilon)^{\frac{\eta}{2}-1}r^{(\lambda^{(m,n)}(g)+\varepsilon)}+O(1). \tag{29}$$

Case II. Let $m > q+1$, $n \geq l$ and $A > \lambda^{(m,n)}(g)$.

In view of (24), we obtain for all sufficiently large values of r that

$$\log^{[p+m-q]} \mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}} r\Big)\Big) \leq \log^{[m]} \mu_{g_{\eta-1}}(r) + O(1)$$

$$i.e.,\ \log^{[p+m-q]} \mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}} r\Big)\Big) \leq \log^{[m]} \mu_g(\mu_{f_{\eta-2}}(r)) + O(1)$$

$$i.e.,\ \log^{[p+m-q]} \mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}} r\Big)\Big) \leq (\rho^{(m,n)}(g)+\varepsilon) \log^{[n]}(\mu_{f_{\eta-2}}(r)) + O(1)$$

$$i.e.,\ \log^{[p+m-q]} \mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}} r\Big)\Big) \leq (\rho^{(m,n)}(g)+\varepsilon) \log^{[l]} \mu_{f_{n-2}}(r) + O(1), \quad (30)$$

Applying (30) to continue this process, we have

$$\log^{[p+m-q]} \mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}} r\Big)\Big) \leq (\rho^{(m,n)}(g)+\varepsilon)(\rho^{(l,q)}(f)+\varepsilon) \log^{[q]} \mu_{g_{\eta-3}}(r) + O(1), \quad (31)$$

and so on.

We finally have the following inequality

$$\log^{[p+(m-q)(\frac{\eta}{2}-1)]} \mu_h^{-1}\Big(\mu_{f_\eta}\Big(\Big(\frac{(\alpha-1)(\beta-1)}{(\alpha-\beta+1)\beta}\Big)^{\frac{\eta}{2}} r\Big)\Big) \leq \log^{[m]} \mu_g(r) + O(1)$$

$$i.e.,\ \log^{[p+(m-q)(\frac{\eta}{2}-1)]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) \leq \log^{[m]} \mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}} \exp^{[n-1]} r\Big) + O(1).$$

Now from (22) and above, it follows for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p+(m-q)(\frac{\eta}{2}-1)]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) &\leq \log r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)} + O(1) \\ i.e.,\ \log^{[p+(m-q)(\frac{\eta}{2}-1)-1]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) &\leq r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)} + O(1). \end{aligned} \quad (32)$$

Now if $q \geq m$, $n = l$ and $A > 1$, we get from (21) and (28) of Case I for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n]} r))} \geq$$

$$\geq \frac{(\lambda_h^{(p,q)}(f)-\varepsilon) r^A}{(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f)+\varepsilon)^{\frac{\eta}{2}-1}(\lambda^{(m,n)}(g)+\varepsilon) r + O(1)},$$

from which the first part of the theorem follows.

Again combining (21) and (29), we obtain for a sequence of values of r tending to infinity when $q \geq m$, $n = l$ and $\lambda^{(m,n)}(g) < A$

$$\frac{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} >$$

$$\frac{(\lambda_h^{(p,q)}(f) - \varepsilon) r^A}{(\rho_h^{(p,q)}(f) + \varepsilon)(\rho^{(m,n)}(g) + \varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f) + \varepsilon)^{\frac{\eta}{2}-1} r^{(\lambda^{(m,n)}(g)+\varepsilon)} + O(1)}. \tag{33}$$

As $A > \lambda^{(m,n)}(g)$ we can choose $\varepsilon(> 0)$ in such a way that

$$\lambda^{(m,n)}(g) + \varepsilon < A. \tag{34}$$

Thus from (33) and (34), we get that

$$\limsup_{r \to +\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty.$$

This establishes the second part of the theorem.

When $m > q + 1$, $n \geq l$ and $A > \lambda^{(m,n)}(g)$, it follows from (21) and (32) of Case III for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p+(m-q)(\frac{\eta}{2}-1)-1]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} \geq \frac{(\lambda_h^{(p,q)}(f) - \varepsilon) r^A}{r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)} + O(1)}. \tag{35}$$

Now from (34) and (35) we obtain that

$$\limsup_{r \to +\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p+(m-q)(\frac{\eta}{2}-1)-1]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty.$$

This proves the third part of the theorem.

Thus the theorem follows. □

In the line of Theorem 5 we may state the following theorem without proof.

Theorem 6. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l,q) and (l,p) respectively such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < +\infty$ and $g(z)$ be an entire function with $0 < \lambda^{(m,n)}(g) \leq \rho^{(m,n)}(g) < +\infty$ and finite relative (p,n)-th lower order $\lambda_k^{(p,n)}(g)$ with respect to another entire function $k(z)$ where l, p, q, m and n are all positive integers. Then*

$$(i) \quad \limsup_{r \to +\infty} \frac{\log^{[p]} \mu_k^{-1}(\mu_g(\exp^{[n]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n]} r))} = +\infty \text{ if } q \geq m, n = l \text{ and } A > 1,$$

$$(ii) \quad \limsup_{r \to +\infty} \frac{\log^{[p]} \mu_k^{-1}(\mu_g(\exp^{[n]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty \text{ if } q = m \text{ or } q \geq m(\neq 1) - 1,$$

$$n \geq l \text{ and } A > \lambda^{(m,n)}(g)$$

and

$$(iii) \limsup_{r\to+\infty} \frac{\log^{[p]} \mu_k^{-1}(\mu_g(\exp^{[n]} r^A))}{\log^{[p+(m-q)(\frac{\eta}{2}-1)-1]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty \text{ if } m > q+1,\ n \geq l \text{ and } A > \lambda^{(m,n)}(g).$$

Theorem 7. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l,q) and (l,p) respectively such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < +\infty$ and $g(z)$ be an entire function with finite (m,n)-th order where l,p,q,m and n are all positive integers. Then*

$$(i) \lim_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n]} r))} = +\infty \text{ if } q \geq m, n = l \text{ and } A > 1,$$

$$(ii) \lim_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty \text{ if } q = m \text{ or } q \geq m(\neq 1)-1,\ n \geq l \text{ and } A > \rho^{(m,n)}(g)$$

and

$$(iii) \lim_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}{\log^{[p+(m-q)(\frac{\eta}{2}-1)-1]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty \text{ if } m > q+1,\ n \geq l \text{ and } A > \rho^{(m,n)}(g).$$

Theorem 8. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l,q) and (l,p) respectively such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < +\infty$ and $g(z)$ be an entire function with $0 < \rho^{(m,n)}(g) < +\infty$ and finite relative (p,n)-th lower order $\lambda_k^{(p,n)}(g)$ with respect to another entire function $k(z)$ where l,p,q,m and n are all positive integers. Then*

$$(i) \lim_{r\to+\infty} \frac{\log^{[p]} \mu_k^{-1}(\mu_g(\exp^{[n]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n]} r))} = +\infty \text{ if } q \geq m, n = l \text{ and } A > 1,$$

$$(ii) \lim_{r\to+\infty} \frac{\log^{[p]} \mu_k^{-1}(\mu_g(\exp^{[n]} r^A))}{\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty \text{ if } q = m \text{ or } q \geq m(\neq 1)-1,\ n \geq l \text{ and } A > \rho^{(m,n)}(g)$$

and

$$(iii) \lim_{r\to+\infty} \frac{\log^{[p]} \mu_k^{-1}(\mu_g(\exp^{[n]} r^A))}{\log^{[p+(m-q)(\frac{\eta}{2}-1)-1]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))} = +\infty \text{ if } m > q+1,\ n \geq l \text{ and } A > \rho^{(m,n)}(g).$$

We omit the proof of Theorem 7 and Theorem 8 as those can be carried out in the line of Theorem 5 and Theorem 6 respectively.

As an application of Theorem 1 and Theorem 5, we may state the following theorem:

Theorem 9. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l, q) and (l, p) respectively such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < +\infty$ and $g(z)$ be an entire function with $\lambda^{(m,n)}(g) < A < \rho^{(m,n)}(g)$ where l, p, q, m and n are all positive integers. Then for $q = m\,(\neq 1) - 1$ and $n = l$.*

$$\limsup_{r\to+\infty} \frac{\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{\mu_h^{-1}(\mu_f(\exp^{[q]} r^A))} \leq 1 \leq \liminf_{r\to+\infty} \frac{\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{\mu_h^{-1}(\mu_f(\exp^{[q]} r^A))}.$$

Proof. In view of Theorem 1 we get for a sequence of values of r tending to infinity and for $K > 1$

$$\begin{aligned}
\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) &> K \log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A)) \\
i.e.,\ \log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) &> \log\left\{\log^{[p-1]} \mu_h^{-1}(\mu_f(\exp^{[q]}\left(r^A\right)))\right\}^K \\
i.e.,\ \log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) &> \log\left\{\log^{[p-1]} \mu_h^{-1}(\mu_f(\exp^{[q]}\left(r^A\right)))\right\}
\end{aligned}$$

$$\begin{aligned}
i.e.,\ \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) &> \mu_h^{-1}(\mu_f(\exp^{[q]}\left(r^A\right))) \\
i.e.,\ \frac{\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{\mu_h^{-1}(\mu_f(\exp^{[q]}(r^A)))} &> 1 \\
i.e.,\ \limsup_{r\to+\infty} \frac{\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r))}{\mu_h^{-1}(\mu_f(\exp^{[q]}(r^A)))} &\geq 1. \qquad (36)
\end{aligned}$$

Again from Theorem 5 we obtain for a sequence of values of r tending to infinity and for $P > 1$

$$\begin{aligned}
\log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A)) &> P \log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r) \\
i.e.,\ \log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A)) &> \log\left\{\log^{[p-1]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)\right\}^P \\
i.e.,\ \log^{[p]} \mu_h^{-1}(\mu_f(\exp^{[q]} r^A)) &> \log\left\{\log^{[p-1]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)\right\}
\end{aligned}$$

$$\begin{aligned}
i.e.,\ \mu_h^{-1}(\mu_f(\exp^{[q]} r^A)) &> \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r) \\
i.e.,\ \frac{\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)}{\mu_h^{-1}(\mu_f(\exp^{[q]} r^A))} &< 1 \\
i.e.,\ \liminf_{r\to+\infty} \frac{\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)}{\mu_h^{-1}(\mu_f(\exp^{[q]} r^A))} &\leq 1. \qquad (37)
\end{aligned}$$

Thus the theorem follows from (36) and (37). □

In view of Theorem 3 and Theorem 6, the following theorem can be carried out:

Theorem 10. *Let $f(z)$ and $h(z)$ be any two entire functions with index-pairs (l,q) and (l,p) respectively such that $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<+\infty$ and $g(z)$ be an entire function with $\lambda^{(m,n)}(g)<A<\rho^{(m,n)}(g)$ and finite relative (p,n)-th order $\rho_k^{(p,n)}(g)$ and relative (p,n)-th lower order $\lambda_k^{(p,n)}(g)$ with respect to another entire function $k(z)$ where l,p,q,m and n are all positive integers. Then for $q=m\,(\neq 1)-1$ and $n=l$.*

$$\limsup_{r\to+\infty}\frac{\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r)}{\mu_k^{-1}(\mu_g(\exp^{[n]}(r^A)))}\leq 1\leq\liminf_{r\to+\infty}\frac{\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r)}{\mu_k^{-1}(\mu_g(\exp^{[n]}(r^A)))}.$$

The proof is omitted.

Theorem 11. *Let $l(z)$, $f(z)$ and $h(z)$ be any three entire functions with index-pairs (c,d), (c,q) and (c,p) respectively such that $\lambda_h^{(p,d)}(l)>0$ and $\rho_h^{(p,q)}(f)<+\infty$. Also let $g(z)$ and $k(z)$ be two entire functions with $\rho^{(m,n)}(g)<\lambda^{(a,b)}(k)$ where a, b, c, d, m, n, p and q are all positive integers.*

$$(i)\ \lim_{r\to+\infty}\frac{\log^{[p-1]}\mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]}r))}{\log^{[p-1]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))\cdot\log^{[p-1]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r))}=+\infty$$

if $d\leq a-1$, $b\leq c$, $q\geq m$ and $n=l$,

$$(ii)\ \lim_{r\to+\infty}\frac{\log^{[p-1]}\mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]}r))}{\log^{[p+(m-q)(\frac{\eta}{2}-1)]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))\cdot\log^{[p-1]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r))}=+\infty$$

if $d\leq a-1$, $b\leq c$, $m-q=2$ and $n\geq l$

and

$$(iii)\ \lim_{r\to+\infty}\frac{\log^{[p-1]}\mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]}r))}{\log^{[p+(m-q)\frac{\eta}{2}-2]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))\cdot\log^{[p-1]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r))}=+\infty$$

if $d\leq a-1$, $b\leq c$, $m-q>2$ and $n\geq l$.

Proof. From the definition of (m,n)-th order of $g(z)$ in terms of maximum terms and for $\alpha>1$, $0<\beta<\alpha$,, we get for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[m]}\mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]}r\Big)\leq$$

$$(\rho^{(m,n)}(g)+\varepsilon)\log^{[n]}\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]}r\Big)$$

$$i.e.,\ \log^{[m]}\mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]}r\Big)\ \leq\ \Big(\rho^{(m,n)}(g)+\varepsilon\Big)\log r+O(1)$$

$$i.e.,\ \log^{[m]}\mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]}r\Big) \leq \log r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1) \quad (38)$$

$$i.e.,\ \log^{[m-1]}\mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}}\exp^{[n-1]}r\Big) \leq r^{(\rho^{(m,n)}(g)+\varepsilon)}+O(1). \quad (39)$$

Also from the definition of (a,b)-th lower order of $k(z)$ in terms of maximum terms, we get for all sufficiently large values of r that

$$\log^{[a]}\mu_k\Big(\frac{\exp^{[b-1]}r}{(196)^{\frac{\eta}{2}}}\Big) \geqslant (\lambda^{(a,b)}(k)-\varepsilon)\log^{[b]}\Big(\frac{\exp^{[b-1]}r}{(196)^{\frac{\eta}{2}}}\Big)$$

$$i.e.,\ \log^{[a]}\mu_k\Big(\frac{\exp^{[b-1]}r}{(196)^{\frac{\eta}{2}}}\Big) \geqslant \log r^{(\lambda^{(a,b)}(k)-\varepsilon)}+O(1) \quad (40)$$

$$i.e.,\ \log^{[a-1]}\mu_k\Big(\frac{\exp^{[b-1]}r}{(196)^{\frac{\eta}{2}}}\Big) \geqslant r^{(\lambda^{(a,b)}(k)-\varepsilon)}+O(1). \quad (41)$$

Again from the definition of (p,q)-th relative order of $f(z)$ with respect to $h(z)$ in terms of maximum terms, we have for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r)) \leq (\rho_h^{(p,q)}(f)+\varepsilon)\log r$$

$$i.e.,\ \log^{[p-1]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r)) \leq r^{(\rho_h^{(p,q)}(f)+\varepsilon)}. \quad (42)$$

Case I. Let $d \leq a-1$ and $b \leq c$.

Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 3 and in view of (4), for all sufficiently large values r that

$$\log^{[p]}\mu_h^{-1}(\mu_{l_\xi}((196)^{\frac{n}{2}}r)) \geq \log^{[p]}\mu_h^{-1}(\mu_l(\mu_{k_{\xi-1}}(r)))$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{l_\xi}((196)^{\frac{n}{2}}r)) \geq (\lambda_h^{(p,d)}(l)-\varepsilon)\log^{[d]}\mu_{k_{\xi-1}}(r)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{l_\xi}((196)^{\frac{n}{2}}r)) > (\lambda_h^{(p,d)}(l)-\varepsilon)\log^{[a]}\mu_k(\mu_{l_{\xi-2}}(r))$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{l_\xi}((196)^{\frac{n}{2}}r)) > (\lambda_h^{(p,d)}(l)-\varepsilon)(\lambda^{(a,b)}(k)-\varepsilon)\log^{[b]}\mu_{l_{\xi-2}}(r)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{l_\xi}((196)^{\frac{n}{2}}r)) > (\lambda_h^{(p,d)}(l)-\varepsilon)(\lambda^{(a,b)}(k)-\varepsilon)\log^{[c]}\mu_{l_{\xi-2}}(r). \quad (43)$$

Applying (43) to continue this process, we have

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{l_\xi}((196)^{\frac{n}{2}}r)) > (\lambda_h^{(p,d)}(l)-\varepsilon)(\lambda^{(a,b)}(k)-\varepsilon)(\lambda^{(c,d)}(l)-\varepsilon)\log^{[d]}\mu_{k_{\xi-3}}(r),$$

and so on.

We finally have the following inequality for all sufficiently large values of r,

$$\log^{[p]}\mu_h^{-1}(\mu_{l_\xi}((196)^{\frac{n}{2}}r)) >$$
$$(\lambda_h^{(p,d)}(l)-\varepsilon)(\lambda^{(a,b)}(k)-\varepsilon)^{\frac{\xi}{2}-1}(\lambda^{(c,d)}(l)-\varepsilon)^{\frac{\xi}{2}-1}\log^{[d]}\mu_k(r).$$

Now from above we get for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]} r)) >$$

$$(\lambda_h^{(p,d)}(l) - \varepsilon)(\lambda^{(a,b)}(k) - \varepsilon)^{\frac{\xi}{2}-1}(\lambda^{(c,d)}(l) - \varepsilon)^{\frac{\xi}{2}-1} \log^{[a-1]} \mu_k\Big(\frac{\exp^{[b-1]} r}{(196)^{\frac{\eta}{2}}}\Big). \quad (44)$$

Now we get from (41) and (44) for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]} r)) >$$

$$(\lambda_h^{(p,d)}(l) - \varepsilon)(\lambda^{(a,b)}(k) - \varepsilon)^{\frac{\xi}{2}-1}(\lambda^{(c,d)}(l) - \varepsilon)^{\frac{\xi}{2}-1} r^{(\lambda^{(a,b)}(k)-\varepsilon)} + O(1)$$

$i.e.,$ $\log^{[p-1]} \mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]} r)) >$

$$\exp((\lambda_h^{(p,d)}(l)-\varepsilon)(\lambda^{(a,b)}(k)-\varepsilon)^{\frac{\xi}{2}-1}(\lambda^{(c,d)}(l)-\varepsilon)^{\frac{\xi}{2}-1} r^{(\lambda^{(a,b)}(k)-\varepsilon)} + O(1)). \quad (45)$$

Case II. Let $q \geq m$ and $n = l$.

Since $q \geq m$, therefore $q \geq m - 1$. Now we obtain in view of (27) and (39) for all sufficiently large values of r

$$\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) \leq$$

$$(\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f)+\varepsilon)^{\frac{\eta}{2}-1} \log^{[m-1]} \mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}} \exp^{[n-1]} r\Big)$$

$i.e.,$ $\log^{[p]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) \leq$

$$(\rho_h^{(p,q)}(f) + \varepsilon)(\rho^{(m,n)}(g) + \varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f) + \varepsilon)^{\frac{\eta}{2}-1} r^{(\rho^{(m,n)}(g)+\varepsilon)} + O(1)$$

$i.e.,$ $\log^{[p-1]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) \leq$

$$\exp((\rho_h^{(p,q)}(f) + \varepsilon)(\rho^{(m,n)}(g) + \varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f) + \varepsilon)^{\frac{\eta}{2}-1} r^{(\rho^{(m,n)}(g)+\varepsilon)} + O(1)). \quad (46)$$

Case III. Let $q < m$ and $n \geq l$.

In view of (31) and (38), we derived the following inequality for all sufficiently large values of r that

$$\log^{[p+(m-q)(\frac{\eta}{2}-1)]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) \leq$$

$$\exp^{[m-q]} \log^{[m]} \mu_g\Big(\Big(\frac{(\alpha-\beta+1)\beta}{(\alpha-1)(\beta-1)}\Big)^{\frac{\eta}{2}} \exp^{[n-1]} r\Big) + O(1)$$

$$i.e., \log^{[p+(m-q)(\frac{\eta}{2}-1)]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) \leq \exp^{[m-q-1]} r^{(\rho^{(m,n)}(g)+\varepsilon)} + O(1). \quad (47)$$

Now if $m - q = 2$, then we get from (47) for all sufficiently large values of r that

$$\log^{[p+(m-q)(\frac{\eta}{2}-1)]} \mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]} r)) \leq \exp r^{(\rho^{(m,n)}(g)+\varepsilon)} + O(1). \quad (48)$$

Also if $m - q > 2$, then we get from (47) for all sufficiently large values of r that

$$\begin{aligned} & \log^{[m-q-2]}\left[\log^{[p+(m-q)(\frac{\eta}{2}-1)]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))\right] \\ \leq \ & \log^{[m-q-2]}\left[\exp^{[m-q-1]}r^{(\rho^{(m,n)}(g)+\varepsilon)}+O(1)\right] \end{aligned}$$

$$i.e., \ \log^{[p+(m-q)\frac{\eta}{2}-2]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r)) \leq \exp r^{(\rho^{(m,n)}(g)+\varepsilon)}+O(1). \tag{49}$$

Now as $\rho^{(m,n)}(g) < \lambda^{(a,b)}(k)$, we can choose $\varepsilon(>0)$ in such a manner that

$$\rho^{(m,n)}(g)+\varepsilon < \lambda^{(a,b)}(k)-\varepsilon. \tag{50}$$

Therefore combining (42), (45) of Case I and (46) of Case II it follows for all sufficiently large values of r that

$$\frac{\log^{[p-1]}\mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]}r))}{\log^{[p-1]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))\cdot\log^{[p-1]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r))} >$$

$$\frac{\exp((\lambda_h^{(p,d)}(l)-\varepsilon)(\lambda^{(a,b)}(k)-\varepsilon)^{\frac{\xi}{2}-1}(\lambda^{(c,d)}(l)-\varepsilon)^{\frac{\xi}{2}-1}r^{(\lambda^{(a,b)}(k)-\varepsilon)}+O(1))}{r^{(\rho_h^{(p,q)}(f)+\varepsilon)}\cdot\exp((\rho_h^{(p,q)}(f)+\varepsilon)(\rho^{(m,n)}(g)+\varepsilon)^{\frac{\eta}{2}-1}(\rho^{(l,q)}(f)+\varepsilon)^{\frac{\eta}{2}-1}r^{(\rho^{(m,n)}(g)+\varepsilon)}+O(1))}.$$

Thus in view of (50) first part of the theorem follows from above.

Again combining (42), (45) of Case I, (48) of Case III and (50) we obtain for all sufficiently large values of r that

$$\frac{\log^{[p-1]}\mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]}r))}{\log^{[p+(m-q)(\frac{\eta}{2}-1)]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))\cdot\log^{[p-1]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r))} >$$

$$\frac{\exp((\lambda_h^{(p,d)}(l)-\varepsilon)(\lambda^{(a,b)}(k)-\varepsilon)^{\frac{\xi}{2}-1}(\lambda^{(c,d)}(l)-\varepsilon)^{\frac{\xi}{2}-1}r^{(\lambda^{(a,b)}(k)-\varepsilon)}+O(1))}{r^{(\rho_h^{(p,q)}(f)+\varepsilon)}\cdot[\exp r^{(\rho^{(m,n)}(g)+\varepsilon)}+O(1)]}$$

$$i.e., \ \lim_{r\to+\infty}\frac{\log^{[p-1]}\mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]}r))}{\log^{[p+(m-q)(\frac{\eta}{2}-1)]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))\cdot\log^{[p-1]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r))} = +\infty,$$

which is the second part of the theorem.

Similarly combining (42), (45) of Case I, (49) of Case III and (50) we get for all sufficiently large values of r that

$$\frac{\log^{[p-1]}\mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]}r))}{\log^{[p+(m-q)\frac{\eta}{2}-2]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))\cdot\log^{[p-1]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r))} >$$

$$\frac{\exp((\lambda_h^{(p,d)}(l)-\varepsilon)(\lambda^{(a,b)}(k)-\varepsilon)^{\frac{\xi}{2}-1}(\lambda^{(c,d)}(l)-\varepsilon)^{\frac{\xi}{2}-1}r^{(\lambda^{(a,b)}(k)-\varepsilon)}+O(1))}{r^{(\rho_h^{(p,q)}(f)+\varepsilon)}\cdot\left[\exp r^{(\rho^{(m,n)}(g)+\varepsilon)}+O(1)\right]}$$

$$i.e., \ \lim_{r\to+\infty}\frac{\log^{[p-1]}\mu_h^{-1}(\mu_{l_\xi}((\exp^{[b-1]}r))}{\log^{[p+(m-q)\frac{\eta}{2}-2]}\mu_h^{-1}(\mu_{f_\eta}(\exp^{[n-1]}r))\cdot\log^{[p-1]}\mu_h^{-1}(\mu_f(\exp^{[q-1]}r))} = +\infty.$$

This proves the third part of the theorem.

Thus the theorem follows. □

Remark 1. If we consider $\rho^{(m,n)}(g) < \rho^{(a,b)}(k)$ instead of $\rho^{(m,n)}(g) < \lambda^{(a,b)}(k)$ and the other conditions remain the same, the conclusion of Theorem 11 remains valid with "limit superior" replaced by "limit".

Remark 2. The same results of above theorems and remarks in terms of maximum modulus of entire functions can also be deduced with the help of Lemma 1 and Lemma 4.

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