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Graphs, Disjoint Matchings and Some Inequalities

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Abstract. A graph G is k-edge-colorable if the edges of G can be assigned a color from $\{1, ..., k\}$ so that adjacent edges of G receive different colors. A maximum k-edge-colorable subgraph of G is a k-edge-colorable subgraph of G containing maximum number of edges. For $k \geq 1$ and a graph G, let $\nu_k(G)$ denote the number of edges in a maximum k-edge-colorable subgraph of G. In 2010 Mkrtchyan, Petrosyan and Vardanyan proved that if G is a cubic graph, then $\nu_2(G) \leq \frac{|V(G)|+2\cdot\nu_3(G)}{4}$ [13]. For cubic graphs containing a perfect matching, in particular, for bridgeless cubic graphs, this inequality can be stated as $\nu_2(G) \leq \frac{\nu_1(G)+\nu_3(G)}{2}$. One may wonder whether there are other well-known graph classes, where a similar result can be obtained. In this work, we prove lower bounds for $\nu_k(G)$ in terms of $\frac{\nu_{k-1}(G)+\nu_{k+1}(G)}{2}$ for $k \geq 2$ and graphs G containing at most 1 cycle. We also present the corresponding conjectures for nearly bipartite graphs.

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1 Introduction

In this paper graphs are assumed to be finite, undirected and without loops, though they may contain multi-edges. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. The degree of a vertex u of G is denoted by $d_G(u)$. Let $\Delta(G)$ be the maximum degree of a vertex of G. A graph is *cubic* if every vertex has degree 3.

A *matching* in a graph is a set of edges without common vertices. A matching which covers all vertices of the graph is called a *perfect matching*. A tree is a connected graph that does not contain a cycle. In the paper, we will consider trees as rooted. Let T be a tree and let r be a vertex of T. We will call r the root of T. Now, let u be any vertex of T. We will say that a vertex v of T is a child of u if v is adjacent to u and it does not lie on the unique path of T connecting u and r. A vertex w of T is called grand-child of u if w is a child of a child of u. Similarly, one can define the notion of a grand-grand-child, etc.

A graph G is called *k-edge colorable* if its edges can be assigned k colors so that adjacent edges receive different colors. A subgraph H of a graph G is called *maximum k-edge-colorable* if H is k-edge-colorable and contains maximum number of edges among all k-edge-colorable graphs. If H is a k-edge-colorable subgraph of G and $e \notin E(H)$, then we will say that e is an *uncolored* edge with respect to H. If

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it is clear from the context with respect to which subgraph an edge is uncolored, we will not mention the subgraph.

By a classical result due to Shannon [17, 20, 22], we have that cubic graphs are 4-edge-colorable. It is an interesting and useful problem to investigate the sizes of subgraphs of cubic graphs that are colorable only with 1, 2 or 3 colors.

For $k \geq 1$ and a graph G let

$$\nu_k(G) = \max\{|E(H)| : H \text{ is a } k\text{-edge-colorable subgraph of } G\}.$$

Albertson and Haas [1, 2], Steffen [18, 19] and Mkrtchyan et al.[13] investigated the lower bounds for $\frac{\nu_k(G)}{|V(G)|}$ in cubic graphs. As a result, in [13] an interesting relation between $\nu_2(G)$ and $\nu_3(G)$ is proved, which states that for any cubic graph G

$$\nu_2(G) \leq \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

Observe that when G contains a perfect matching ($\nu_1(G) = \frac{|V(G)|}{2}$), in particular, when G is a bridgeless cubic graph, the above-mentioned inequality can be written as

$$\nu_2(G) \leq \frac{\nu_1(G) + \nu_3(G)}{2}.$$

One may wonder whether a bound for $\nu_2(G)$ can be proved in terms of $\frac{\nu_1(G)+\nu_3(G)}{2}$ in other interesting graph classes. In the present work we investigate the problem in nearly bipartite graphs. Recall that a graph G is bipartite if $V(G)$ can be partitioned into two sets V_1 and V_2 such that any edge of G joins a vertex from V_1 to a vertex from V_2. G is nearly bipartite if G contains a vertex w such that $G - w$ is bipartite. Our conjecture states:

***Conjecture* 1.** *For any $k \geq 2$ and a nearly bipartite graph G,*

$$\nu_k(G) \geq \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

Let us note that in [12], the following result is obtained for bipartite graphs:

***Theorem* 1.** *[12] For any $k \geq 2$, $i = 1, ..., k$, and a bipartite graph G,*

$$\nu_k(G) \geq \frac{\nu_{k-i}(G) + \nu_{k+i}(G)}{2}.$$

This theorem amounts to saying that the sequence ν_k is convex in the class of bipartite graphs. Our main result states that Conjecture 1 is true for graphs G containing at most 1 cycle. Let us note that in [12], the following general conjecture is presented, which if true, would imply Conjecture 1:

Conjecture 2. *[12] For any $k \geq 2$, $i = 1, ..., k$, and a graph G,*

$$\nu_k(G) \geq \frac{\nu_{k-i}(G) + \nu_{k+i}(G) - b(G)}{2}.$$

Finally, let us note that the lower bounds for $\frac{\nu_k(G)}{|V(G)|}$ in cubic graphs has been investigated in [4,9,14,15,23] when $k = 1$, and for regular graphs of high girth in [6]. This lower bounds has also been investigated in the case when the graphs need not be cubic [7,11,16].

Terms and concepts that we do not define, can be found in [8,24].

2 The main result

In this section, we prove some lemmas that will be helpful later in the section. Then we verify Conjecture 1 for unicyclic graphs (graphs containing exactly 1 cycle).

Lemma 1. *Let G be a graph, and let $e = (u, v) \in E(G)$. Assume that $d_G(u) = 1$. Then for any $k \geq 1$, there is a maximum k-edge-colorable subgraph H_k of G such that $e \in E(H_k)$.*

Proof. Let H_k be any maximum k-edge-colorable subgraph of G. If $e \in E(H_k)$, then we are done. Thus, we can assume that $e \notin E(H_k)$. Since H_k is a maximum k-edge-colorable subgraph of G and $d_G(u) = 1$, there is an edge $e' \in E(H_k)$ such that e' is incident to v. Consider the subgraph H'_k of G defined as follows: $E(H'_k) = (E(H_k) \backslash \{e'\}) \cup \{e\}$. Observe that H'_k is k-edge-colorable, $e \in E(H'_k)$ and $|E(H'_k)| = |E(H_k)|$, hence H'_k is a maximum k-edge-colorable subgraph of G containing e. □

Lemma 2. *Let $k \geq 1$, G be a connected graph, and let $e = (u, v) \in E(G)$ be a bridge of G. Assume that there is a maximum k-edge-colorable subgraph H_k of G such that $e \in E(H_k)$. Then*

$$\nu_k(G) = \nu_k(G_1e) + \nu_k(G_2e) - 1.$$

Here G_1 and G_2 are the components of $G - e$, and G_1e, G_2e are the supergraphs of G_1 and G_2, respectively, that satisfy the equalities $E(G_1e) = E(G_1) \cup \{e\}$ and $E(G_2e) = E(G_2) \cup \{e\}$.

Proof. Let $H^{(1)}$ and $H^{(2)}$ be the restrictions of H_k in the graphs G_1e and G_2e, respectively. Clearly, these subgraphs are k-edge-colorable. We claim that $H^{(1)}$ and $H^{(2)}$ are maximum k-edge-colorable subgraphs of G_1e and G_2e, respectively. Assume that $|E(H^{(1)})| < \nu_k(G_1e)$. Then, by Lemma 1, there is a maximum k-edge-colorable subgraph $H'^{(1)}$ containing e. Consider the subgraph H'_k of G defined as follows:

$$E(H'_k) = (E(H_k) \backslash E(H^{(1)})) \cup E(H'^{(1)}).$$

Observe that H'_k is k-edge-colorable and $|E(H'_k)| > |E(H_k)|$ contradicting the choice of H_k. Similarly, one can prove that $H^{(2)}$ is a maximum k-edge-colorable subgraphs of G_2e.

We have the following chain of equalities:

$$\nu_k(G) = |E(H_k)| = |E(H^{(1)})| + |E(H^{(2)})| - 1 = \nu_k(G_1 e) + \nu_k(G_2 e) - 1.$$

□

Our first theorem verifies Conjecture 1 for connected graphs with at most 1 cycle.

Theorem 2. *For any $k \geq 2$ and a connected graph G containing at most 1 cycle,*

$$\nu_k(G) \geq \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

Proof. Assume that the statement of the theorem is wrong. Consider all possible counter-examples G and among them choose one minimizing $|V(G)| + |E(G)|$. Clearly, $|V(G)| \geq 3$. Moreover, $\Delta(G) \geq 3$ (the statement of the theorem is true for cycles and paths).

Let T be a tree defined as follows: if G is a tree, then $T = G$, otherwise $T = G/C$. Here C is the only cycle of G, and T is the tree obtained from G by contracting C to a vertex v_C. View T as a rooted tree. The root of T is any of its vertices, if $G = T$, and is the vertex v_C, otherwise. Below, we will speak about children, grand-children of vertices of G. This relationship will be viewed from the perspective of the tree T.

Let us show that there is no vertex of G with degree 2 that is adjacent to a vertex of degree 1. On the opposite assumption, consider a vertex z of degree 2 that is adjacent to a vertex y of degree 1. Observe that since $k \geq 2$, we have $\nu_i(G) = 1 + \nu_i(G - y)$ for $i = k, k+1$ and $\nu_{k-1}(G) \leq 1 + \nu_{k-1}(G - y)$. Thus, we will have:

$$\nu_k(G) = \nu_k(G-y) + 1 \geq \left\lfloor \frac{\nu_{k-1}(G-y) + \nu_{k+1}(G-y)}{2} \right\rfloor + 1$$
$$= \left\lfloor \frac{\nu_{k-1}(G-y) + 1 + \nu_{k+1}(G-y) + 1}{2} \right\rfloor \geq \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

Here we used the fact that $G - y$ is not a counter-example to our statement. Thus, there is no vertex of G that has degree 2 is adjacent to a vertex of degree 1.

Next, let us show that all vertices of G with degree at least 3 lie on C, the unique cycle of G. On the opposite assumption, consider a vertex x of degree at least 3 that does not lie on the cycle and it has no children, grand-children, etc. that are of degree at least 3. Observe that all the children of x are of degree 1. We will consider some cases.

Case 1: $d_G(x) \geq k+2$. Then G can be represented as in Figure 1.

It can be easily seen that in this case there is an edge e adjacent to x such that $\nu_i(G) = \nu_i(G - e)$ for $i = k-1, k, k+1$. Hence, we have:

$$\nu_k(G) = \nu_k(G-e) \geq \left\lfloor \frac{\nu_{k-1}(G-e) + \nu_{k+1}(G-e)}{2} \right\rfloor = \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

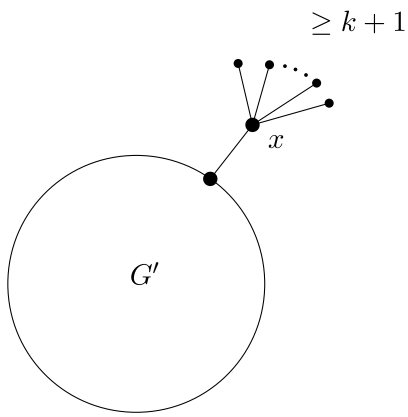

Figure 1. $d_G(x) \geq k+2$

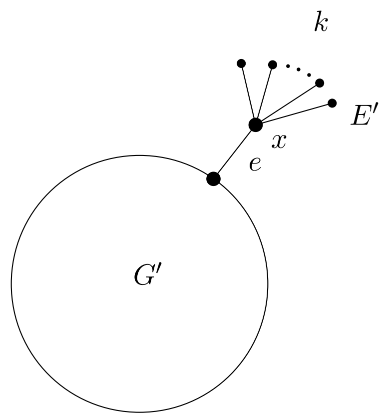

Figure 2. $d_G(x) = k+1$

Here we used the fact that $G - e$ is not a counter-example to our statement.

Case 2: $3 \leq d_G(x) = k+1$. Then G can be represented as in Figure 2. Here E' denotes the edge-set of the component of $G - e$ containing x.

We have

$$\nu_{k-1}(G) \leq \nu_{k-1}(G') + |E'| - 1,$$

$$\nu_k(G) = \nu_k(G') + |E'|,$$

$$\nu_{k+1}(G) = \nu_{k+1}(G'e) + |E'|.$$

It is easy to see that $\nu_{k+1}(G'e) \leq \nu_{k+1}(G') + 1$. Since G' is not a counter-example to our statement, we have

$$\left\lfloor \frac{\nu_{k-1}(G') + \nu_{k+1}(G'e) - 1}{2} \right\rfloor \leq \left\lfloor \frac{\nu_{k-1}(G') + \nu_{k+1}(G')}{2} \right\rfloor \leq \nu_k(G').$$

The last inequality, in its turn, implies:

$$\nu_k(G) = \nu_k(G') + |E'| \geq \left\lfloor \frac{\nu_{k-1}(G') + \nu_{k+1}(G'e) - 1}{2} \right\rfloor + |E'| \geq \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

Case 3: $3 \leq d_G(x) = k$. Then G can be represented as in Figure 3. Here E' denotes the edge-set of the component of $G - e$ containing x.

We have the following equalities:

$$\nu_{k-1}(G) = \nu_{k-1}(G') + |E'|,$$

$$\nu_k(G) = \nu_k(G'e) + |E'|,$$

$$\nu_{k+1}(G) = \nu_{k+1}(G'e) + |E'|.$$

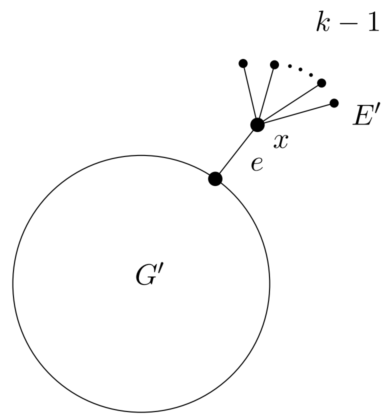

Figure 3. $d_G(x) = k$

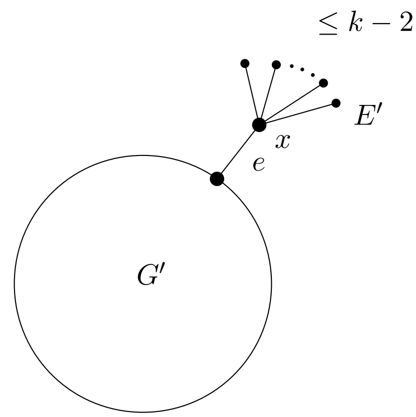

Figure 4. $d_G(x) \leq k-1$

It is easy to see that $\nu_{k-1}(G') \leq \nu_{k-1}(G'e)$. Since $G'e$ is not a counter-example, we have

$$\left\lfloor \frac{\nu_{k-1}(G') + \nu_{k+1}(G'e)}{2} \right\rfloor \leq \left\lfloor \frac{\nu_{k-1}(G'e) + \nu_{k+1}(G'e)}{2} \right\rfloor \leq \nu_k(G'e).$$

The last inequality, in turn, implies:

$$\nu_k(G) = \nu_k(G'e) + |E'| \geq \left\lfloor \frac{\nu_{k-1}(G') + \nu_{k+1}(G'e)}{2} \right\rfloor + |E'| = \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

Case 4: $3 \leq d_G(x) \leq k-1$. Then G can be represented as in Figure 4. Here E' denotes the edge-set of the component of $G - e$ containing x. We have the following equalities:

$$\nu_{k-1}(G) = \nu_{k-1}(G'e) + |E'|,$$
$$\nu_k(G) = \nu_k(G'e) + |E'|,$$
$$\nu_{k+1}(G) = \nu_{k+1}(G'e) + |E'|.$$

Since $G'e$ is not a counter-example, we have

$$\nu_k(G'e) \geq \left\lfloor \frac{\nu_{k-1}(G'e) + \nu_{k+1}(G'e)}{2} \right\rfloor.$$

Hence,

$$\nu_k(G) = \nu_k(G'e) + |E'| \geq \left\lfloor \frac{\nu_{k-1}(G'e) + \nu_{k+1}(G'e)}{2} \right\rfloor + |E'|$$
$$= \left\lfloor \frac{\nu_{k-1}(G'e) + |E'| + \nu_{k+1}(G'e) + |E'|}{2} \right\rfloor = \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

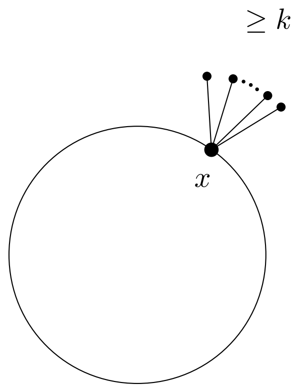

Figure 5. $d_G(x) \geq k+2$

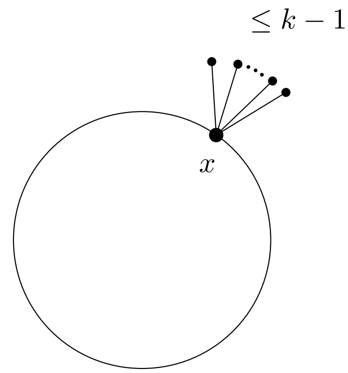

Figure 6. $d_G(x) \leq k+1$

The considered cases imply that all vertices of G with degree at least 3 lie on C. If there is a vertex x of G lying on C with $d_G(x) \geq k+2$, then G can be represented as in Figure 5.

Observe that there is an edge e of C that is incident to x and $\nu_{k+1}(G) = \nu_{k+1}(G-e)$. Moreover, for any edge f of C that is incident to x, $\nu_i(G) = \nu_i(G-f)$ for $i = k-1, k$. Hence we have:

$$\nu_k(G) = \nu_k(G-e) \geq \left\lfloor \frac{\nu_{k-1}(G-e) + \nu_{k+1}(G-e)}{2} \right\rfloor = \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

Here the inequality follows from the fact that $G-e$ is not a counter-example to our statement.

Thus, we can assume that for any vertex x of G lying on C, we have $d_G(x) \leq k+1$. Then G can be represented as in Figure 6.

Let us show that $\nu_{k+1}(G) = |E(G)|$, that is, G is $(k+1)$-edge-colorable. Consider the colors $\{1, 2, ..., k, k+1\}$. Color the edges of the cycle C with colors $1, 2, 3$. Observe that at each vertex of C only two colors will be present. Hence at each vertex of C $k-1$ colors will be missing. Since each vertex x of C is adjacent to at most $k-1$ vertices lying outside C, we can extend the edge-coloring of C, to a $(k+1)$-edge-coloring of G.

Define x_{k-1} and x_k as the minimum number of edges of C that one needs to remove from G in order to obtain a $(k-1)$- or k-edge-colorable subgraph of G, respectively. We have:

$$\nu_{k-1}(G) = |E(G)| - x_{k-1},$$
$$\nu_k(G) = |E(G)| - x_k,$$
$$\nu_{k+1}(G) = |E(G)|.$$

Let us show that

$$x_k \leq \left\lceil \frac{x_{k-1}}{2} \right\rceil. \tag{1}$$

Let J_{k-1} be a subgraph of C such that $G - E(J_{k-1})$ is $(k-1)$-edge-colorable and $|E(J_{k-1})| = x_{k-1}$. Observe that $\Delta(J_{k-1}) \leq 2$, hence

$$\nu_1(J_{k-1}) \geq \left\lfloor \frac{|E(J_{k-1})|}{2} \right\rfloor = \left\lfloor \frac{x_{k-1}}{2} \right\rfloor.$$

Let M_{k-1} be a maximum matching of J_{k-1}. Then $G - (E(J_{k-1}) \backslash M_{k-1})$ is k-edge-colorable, hence

$$x_k \leq |E(J_{k-1}) \backslash M_{k-1}| \leq \left\lceil \frac{|E(J_{k-1})|}{2} \right\rceil = \left\lceil \frac{x_{k-1}}{2} \right\rceil.$$

Finally, let us note that (1) is equivalent to

$$\nu_k(G) \geq \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

Hence G is not a counter-example, which contradicts our assumption. □

***Remark* 1.** *For any $k \geq 2$, there is an infinite sequence of connected graphs G containing one cycle such that*

$$\nu_k(G) = \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

Proof. Let $k \geq 2$ be a fixed integer. For a positive integer $l \geq 2$ consider the graph G from Figure 7. G contains one cycle C_l of length l. Every vertex lying on C_l is of degree $k+1$. It is incident to two edges lying on the cycle and $k-1$ other edges, whose other endvertices are of degree one.

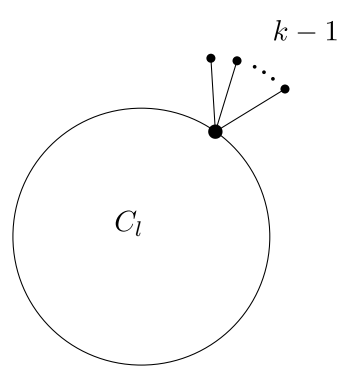

Figure 7. The infinite sequence of graphs.

It can be easily checked that

$$\nu_{k-1}(G) = l \cdot (k-1),$$
$$\nu_k(G) = l \cdot (k-1) + \left\lfloor \frac{l}{2} \right\rfloor,$$
$$\nu_{k+1}(G) = |E(G)| = l \cdot (k-1) + l,$$

hence

$$\nu_k(G) = \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

□

The next theorem follows from Theorem 1.

Theorem 3. *For any $k \geq 2$ and a connected bipartite graph G containing at most 1 cycle,*

$$\nu_k(G) \geq \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}.$$

Corollary 1. *For any $k \geq 2$ and a tree T*

$$\nu_k(T) \geq \frac{\nu_{k-1}(T) + \nu_{k+1}(T)}{2}.$$

Combined with the classical theorem of König [24], Corollary 1 implies:

Corollary 2. *If T is a tree containing a perfect matching and $\Delta(T) = 3$, then*

$$\nu_2(T) \geq \frac{3|V(T)| - 2}{4}.$$

Remark 2. *For any $k \geq 2$, there is an infinite sequence of connected bipartite graphs G containing 1 cycle such that*

$$\nu_k(G) = \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}.$$

Proof. Consider the sequence of graphs G from Remark 1 when l is even. Observe that

$$\nu_k(G) = \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2}.$$

□

Our final remark shows that the proved inequalities are true when G need not be connected.

***Remark* 3.** *Let G be a graph containing at most 1 cycle. Then:*

(1) if G is bipartite, then $\nu_k(G) \geq \frac{\nu_{k-1}(G)+\nu_{k+1}(G)}{2}$,

(2) $\nu_k(G) \geq \left\lfloor \frac{\nu_{k-1}(G)+\nu_{k+1}(G)}{2} \right\rfloor$.

Proof. Let G be comprised of t connected components $G^{(1)}$, ..., $G^{(t)}$. Then for $j = k-1, k, k+1$, we have

$$\nu_j(G) = \nu_j(G^{(1)}) + ... + \nu_j(G^{(t)}).$$

Hence if G is bipartite, we have the statement (1). Let us prove (2). We can assume that $G^{(1)}$ contains exactly one odd cycle. Hence $G^{(2)}$, ..., $G^{(t)}$ are trees. Let R be the graph comprised of components $G^{(2)}$, ..., $G^{(t)}$. By (1), we have

$$\nu_k(R) \geq \frac{\nu_{k-1}(R) + \nu_{k+1}(R)}{2}.$$

Also, since $G^{(1)}$ is connected, we have

$$\nu_k(G^{(1)}) \geq \left\lfloor \frac{\nu_{k-1}(G^{(1)}) + \nu_{k+1}(G^{(1)})}{2} \right\rfloor.$$

Thus,

$$\begin{aligned} 2\nu_k(G) &= 2\nu_k(G^{(1)}) + 2\nu_k(R) \geq \nu_{k-1}(G^{(1)}) + \nu_{k+1}(G^{(1)}) - 1 + \nu_{k-1}(R) + \nu_{k+1}(R) \\ &= \nu_{k-1}(G) + \nu_{k+1}(G) - 1, \end{aligned}$$

which is equivalent to

$$\nu_k(G) \geq \left\lfloor \frac{\nu_{k-1}(G) + \nu_{k+1}(G)}{2} \right\rfloor.$$

□

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