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The linear Fredholm integral equations with functionals and parameters

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Abstract. The theory of linear Fredholm integral-functional equations of the second kind with linear functionals and a parameter is considered. The necessary and sufficient conditions are obtained for the coefficients of the equation and those parameter values in the neighbourhood of which the equation has solutions. The leading terms of the asymptotics of the solutions are constructed. The constructive method is proposed for constructing a solution both in the regular case and in the irregular one. In the regular case, the solution is constructed as a Taylor series in powers of the parameter. In the irregular case, the solution is constructed as a Laurent series in powers of the parameter. The example is used to illustrate the proposed constructive theory and method.

Mathematics subject classification: 45B05, 46N20.
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1 Introduction

This paper deals with some issues in the theory of linear integral equations with linear functionals. Modern views on the fundamental laws of nature are often stated in terms of integral equations [1–5]. Many inverse problems in mathematical physics can be formulated or reduced to nonclassical integral equations. In [6] the problem for identification of external force and heat source density dynamics was reduced to solution of Volterra integral equations of the first kind. The analysis of integral operators includes questions of finding eigenvalues and adjoint functions [7], studying the convergence of their asymptotics, existence and convergence theorems of approximate methods [4,5]. At the end of 20th century, A. P. Khromov found a new class of integral operators with discontinuous kernels and began a systematic study of them [2]. Under very general assumptions, he derived the conditions under which eigenfunction expansions of these operators behave like trigonometric Fourier series. However, these conditions as well as the construction of the classical discontinuous Fredholm resolvent in the form of the ratio of two integer analytic expansions over a parameter are difficult to verify. In the works [5, 8, 9] a class of equations with discontinuous kernels was distinguished and studied.

In [10] the branching solutions of the Cauchy problem for nonlinear loaded differential equations with bifurcation parameters were studied. The purpose of this

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study is to prove the properties of the resolvent integral operator as applied to the second kind Fredholm integral equations with local and integral loads, and to formulate and prove constructive theorems of existence and convergence to the desired solution of successive approximations.

Let us consider the equation

$$x - \mathcal{L}x - \lambda\mathcal{K}x = f, \tag{1}$$

where linear operators $\mathcal{L}$ and $\mathcal{K}$ are given as follows

$$\mathcal{L}x := \sum_{k=1}^{n} a_k(t)\langle\gamma_k, x\rangle,$$

$$\mathcal{K}x := \int_a^b K(t,s)x(s)\,ds,$$

λ is a parameter. All the functions in equation (1) are assumed to be continuous. Kernel $K(t,s)$ can be symmetric and it is also continuous both in t and s. The desired solution $x(t)$ is constructed in $\mathcal{C}_{[a,b]}$.

Linear functionals $\langle\gamma_k, x\rangle$ in applications corresponds to the *loads* imposed on the desired solution. The loads can be local ($\langle\gamma_k, x\rangle = x(t_k)$, $t_k \in [a,b]$) or integral such as $\langle\gamma_k, x\rangle = \int_a^b \gamma_k(t)x(t)\,dt$, where $\gamma_k(t)$ are piecewise continuous functions for $t \in [a,b]$ or $\langle\gamma_k, x\rangle = \int_a^b x(t)\,d\gamma_k(t)$, $\gamma_k(t)$ is a given function of limited variation.

The objective is to construct the solution $x(t,\lambda)$ for $\lambda \in \mathbb{R}^1$ of equation (1). For operator $\mathcal{L}x$ below the following brief notation

$$\mathcal{L}x := \sum_{k=1}^{n} a_k(t)\langle\gamma_k, x\rangle \equiv (\vec{a}(t), \langle\vec{\gamma}, x\rangle)$$

is used, where conventional notation $(\cdot,\cdot)$ for scalar product is used. Here $\vec{a}(t) = (a_1(t), \cdots, a_n(t))^T$, $a_i(y) \in \mathcal{C}_{[a,b]}$, $\langle\vec{\gamma}, x\rangle = (\langle\gamma_1, x\rangle, \ldots, \langle\gamma_n, x\rangle)^T$.

Loaded differential equations have been intensively studied during the last decades. The term "loaded equation" was first used in the works of A. M. Nakhushev, here readers may refer to his monograph [3]. Loaded equations appear in many applications, see e.g.[11,12]. But theory and numerical methods for the loaded integral equations remained less developed. In paper [13] the problem statement for the integral equation with single load is given. Then, in [14,15] theory of the Hammerstein integral equations with loads and bifurcation parameters was proposed. In [16] the Fredholm resolvent was employed for computing H_2-norm for linear periodic systems.

The similar statement is addressed in the present paper and analytical method is described which makes it possible to consider integral equations with arbitrary finite number of local and integral loads. An example of functionals that generate local and integral loads in the space $\mathcal{C}_{[a,b]}$ is the functional

$$\langle\gamma, x\rangle := \sum_{i=1}^{m} \alpha_i x(t_i) + \sum_{i=1}^{n} \int_{a_i}^{b_i} m_i(s)x(s)\,ds,$$

where $\alpha_i \in \mathbb{R}^1$, $[a_i, b_i] \subset [a, b]$, $m_i(s) \in \mathcal{C}_{[a_i,b_i]}$, $t_i \in [a, b]$.

2 System of equations to determine the load

Let us introduce the following condition

I. $\langle\gamma_k, K(t,s)\rangle = 0$, $k = 1, \ldots, n$, $s \in [a, b]$ and vectors $\vec{x}_\gamma = (\langle\gamma_1, x\rangle, \ldots, \langle\gamma_n, x\rangle)^T$, $\vec{f}_\gamma = (\langle\gamma_1, f\rangle, \ldots, \langle\gamma_n, f\rangle)^T$.

Lemma 1. *Let condition* **I** *be fuilfilled. Then load vector $\vec{x}_j$ necesserily satisfies system*

$$(E - A_0)\vec{c} = \vec{f}_\gamma, \tag{2}$$

where $A_0 = [\langle\gamma_i, a_k\rangle]_{i,k=1}^n$, E is $(n \times n)$ identity matrix.

Proof. Let us apply the functionals $\langle\gamma_i, \cdot\rangle$, $i = 1, \ldots, n$, to both parts of equation (1). Using **I**, the following system can be derived

$$\langle\gamma_i, x\rangle - \sum_{k=1}^{n}\langle\gamma_i, a_k\rangle\langle\gamma_k, x\rangle = \langle\gamma_i, f\rangle,\ i = 1, \ldots, n. \tag{3}$$

System of linear algebraic equations (3) is, in fact, system (2) presented in coordinate system. Lemma is proved. □

From this Lemma follows:

Corollary 1. *Let condition* **I** *be fuilfilled and system (2) has no solution. Then equation (1) has no solution in class of continuous functions.*

Let condition **I** be fuilfilled and vector $\vec{c}^* \in \mathbb{R}^n$ satisfies system (2). Then solution $x(t, \lambda)$ of equation (1) depends on vector $\vec{c}^*$ and satisfies the following Fredholm integral equation of the 2nd kind

$$x(t, \lambda) - \lambda \int_a^b K(t, s)x(s, \lambda)\, ds = f(t) + (\vec{a}(t), \vec{c}^*).$$

Lemma 2. *Solution of equation (1) for arbitrary λ, except the characteristic numbers λ_i of kernel $K(t, s)$, is defined by the following formula*

$$x(t, \lambda) = (\vec{a}(t), \vec{x}_\gamma(\lambda)) + \int_a^b \Gamma(t, s, \lambda)(\vec{a}(s), \vec{x}_\gamma(\lambda))\, ds + \int_a^b \Gamma(t, s, \lambda)f(s)\, ds + f(t). \tag{4}$$

Here $\Gamma(t, s, \lambda) = \frac{D(t,s,\lambda)}{D(\lambda)}$, $D(t, s, \lambda)$ and $D(\lambda)$ are entire analytic functions of parameter λ, $D(\lambda_i) = 0$. Load vector $\vec{x}_\gamma(\lambda)$ necesserily must satisfy the following system of n linear algebraic equations

$$(E - A_0 - A(\lambda))\vec{x}_\gamma(\lambda) = \vec{b}(\lambda) \tag{5}$$

with matrix

$$A(\lambda) = \left\langle \gamma_i, \int_a^b \Gamma(t,s,\lambda) a_k(s)\, ds \right\rangle_{i,k=1}^{n} \tag{6}$$

and vector

$$\vec{b}(\lambda) = \left\langle \gamma_i, f(t) + \int_a^b \Gamma(t,s,\lambda) f(s)\, ds \right\rangle_{i=1}^{n}.$$

The set of characteristic numbers $\{\lambda_i\}$ *is a finite and countable set .*

Proof. It is known (see sec. 9 (3) in book [17]) that an inverse operator $(I - \lambda K)^{-1}$ is defined by Fredholm formula [18]:

$$(I - \lambda K)^{-1} = I + \lambda \int_a^b \frac{D(t,s,\lambda)}{D(\lambda)} [\cdot]\, ds.$$

Functions $D(t,s,\lambda)$ and $D(\lambda)$ are entire analytical funcations with respect to λ, defined for $\lambda \in \mathbb{R}^1$. Moreover, the characteristic numbers of kernel $K(t,s)$ of operator $\mathcal{K}$ are zeros of denominator $D(\lambda)$. Thus, the inverse operator $(I - \lambda K)^{-1}$ can be called discontinuous operator. Indeed, the function $\Gamma(t,s,\lambda)$ in solution (4) has the 2nd kind discontinuities at points $\{\lambda_i\}$. By solving system (5) and substituting its solution into (4), we find the solution of the original problem (1). The lamma is proved.

□

Remark 1. In system (5) in general case the matrix $A(\lambda)$ and vector $\vec{b}(\lambda)$ will have 2nd kind discontinuities at points λ.

Let us distinguish the class of kernels $K(t,s)$ when matrix A_0 and vector $\vec{b}(\lambda)$ can be specified. Let the kernel $K(t,s)$ generate the nilpotency of the operator $\mathcal{K}$.

Let $|\lambda| < \frac{1}{||\mathcal{K}||}$. In that case the solution of equation $x - \lambda \mathcal{K} x = f$ for arbitrary source function f is defined uniquely as follows

$$x = f + \lambda \mathcal{K} f + \lambda^2 \mathcal{K}^2 f + \cdots + \lambda^p \mathcal{K}^p f.$$

Here

$$\mathcal{K}^n f = \int_a^b K_n(t,s) f(s)\, ds,$$

where

$$K_n(t,s) = \int_a^b K(t,z) K_{n-1}(z,s)\, dz.$$

Here $K_1(t,s) := K(t,s)$, $K_{p+1}(t,s) = 0$ due to the nilpotency of the operator $\mathcal{K}$ for some $p \geq 1$. Therefore, formula (4) can be presented in the following constructive form

$$x(t,\lambda,\vec{x}_\gamma) = f(t) + (\vec{a}(t), \vec{x}_\gamma) + \int_a^b \big(\lambda K(t,s) + \lambda^2 K_2(t,s) + \cdots \tag{7}$$

$$\cdots + \lambda^p K_p(t,s)\big)(f(s) + (\vec{a}(s), \vec{x}_\gamma))\, ds.$$

Correspondingly, we derive the refined system of linear algebraic equations (5) with respect to the load vector because

$$A(\lambda) = \langle \gamma_i, \int_a^b (\lambda K(t,s) + \lambda^2 K_2(t,s) + \cdots + \lambda^p K_p(t,s)) a_k(s)\, ds \rangle_{i,k=1}^n, \quad (8)$$

$$\vec{b}(\lambda) = \langle \gamma_i,\ f(t) + \int_a^b (\lambda K(t,s) + \lambda^2 K_2(t,s) + \cdots + \lambda^p K_p(t,s)) f(s)\, ds \rangle_{i=1}^n. \quad (9)$$

Thus, $A(\lambda)$ and $\vec{b}(\lambda)$ are continuous in λ. It is to be noted that if $\langle \gamma_i, K(t,s)\rangle = 0$, $i = 1, \ldots, n$, then $A(\lambda) = 0$, and system (5) degenerates into system (2) introduced in Lemma 1. Therefore, in this case vector $\vec{x}_\gamma$ from solution (7) to given problem (1) can be determined. Then the following theorem can be formulated.

Theorem 1. *Let operator $\mathcal{K}$ be nilpotent and $\langle \gamma_i, K(t,s)\rangle = 0$, $i = 1, \ldots, n$, $\forall s \in [a,b]$. Then solution of equation (1) exists as functional polynomial (7) of p-th order in parameter λ. Coefficients of polynomial (7) depend on selection of the load vector $\vec{x}_\gamma$ in $\mathbb{R}^n$.*

If operator $\mathcal{K}$ is not nilpotent and the identity $\langle \gamma_i, K(t,s)\rangle = 0$ is not satisfied, then the solution $x(t,\lambda)$ of equation (1) can be found in the class of continuous in t functions. This solution can be represented in the punctured neighbourhood $0 < |\lambda| < \rho$ in the form of Laurent series with pole at point $\lambda = 0$.

3 Successive approximations

Let $\det(E - A_0) \neq 0$. Then there exists a neighbourhood of λ $|\lambda| < \rho$ such that system (5) has a solution $\vec{x}_\rho(\lambda) \to (E - A_0)^{-1}\vec{f}_\gamma$ as $\lambda \to 0$. Positive ρ exists since $||(E - A_0)^{-1}A(\lambda)|| \to 0$ as $\lambda \to 0$.

Let us call the case of $\det(E - A_0) \neq 0$ regular.

Theorem 2. *In the regular case $\det(E - A_0) \neq 0$ there exists a neighbourhood $|\lambda| < \rho$ in which equation (1) has the unique solution continuous in t and holomorphic in λ.*

Corollary 2. *Let $\det(E - A_0) \neq 0$, $||(I-L)^{-1}\mathcal{K}|| \leq l$. Fix the scalar $q < 1$. Then for $|\lambda| \leq \frac{q}{l}$ equation (1) has the unique solution. Moreover, solution is holomorphic in λ. The sequence $\{x_n\}$, where $x_n = \lambda(I-L)^{-1}\mathcal{K}x_{n-1} + (I-L)^{-1f}$, $x_0 = 0$, uniformly converges to the desired solution $x(t,\lambda)$ of equation (1) at the rate of a geometric progression with the denominator $q < 1$.*

Let us focus now on the irregular case of $\det(E - A_0) = 0$. Let $A_0 = E$. Then $\det(E - A_0) = 0$ and we have irregular case. Let $\frac{d^i}{d\lambda^i} A(\lambda)\big|_{\lambda=0}$ for $i = 0, 1, \dots, p-1$ be zero matrices and $\frac{d^p}{d\lambda^p} A(\lambda)\big|_{\lambda=0} \neq 0$. Then the load vector $\vec{x}_\gamma$ satisfies the following system

$$\big(-E - A_p^{-1} \sum_{m=p+1}^{\infty} \lambda^{m-p} A_m\big)\vec{x}_\gamma = \lambda^{-p} A^{-1} \vec{b}(\lambda),$$

where

$$A_p = \frac{1}{p!}\left(\frac{d^p}{d\lambda^p} A(\lambda)\right)\bigg|_{\lambda=0}.$$

Let's select neighbourhood $|\lambda| < \rho$ such that

$$||A_p^{-1} \sum_{m=p+1}^{\infty} \lambda^{m-p} A_m|| \leq q < 1.$$

Then

$$\lambda^p \vec{x}_\gamma = -\sum_{n=0}^{\infty} (-A_p^{-1} \sum_{m=p}^{\infty} \lambda^{m-p} A_m)^n A_p^{-1} \vec{b}(\lambda),$$

which series converges to holomorphic function

$$\vec{\nu}(\lambda) = -\sum_{n=0}^{\infty} (-A_p^{-1} \sum_{m=p}^{\infty} \lambda^{m-p} A_m)^n A_p^{-1} \vec{b}(\lambda)$$

at the rate of a geometric sequence with the denominator $q < 1$ for $|\lambda| \leq \rho$. Therefore, the load $\vec{x}_\gamma(\lambda) = \lambda^{-p} \vec{\nu}(\lambda)$ is a Laurent series with pth order pole. Then the following theorem is true.

Theorem 3. *Let $A_0 = E$, $A(\lambda) = \sum_{m=p}^{\infty} A_m \lambda^m$, $p \geq 1$. Let the matrix A_p be not singular. Then there exists punctured neighbourhood $0 < |\lambda| \leq p$ such that the equation (1) has a solution $x(t, \lambda)$ with pole at point $\lambda = 0$ of order less than or equal to p.*

Example 1. Let us consider the equation

$$x(t, \lambda) - a(t)x(0, \lambda) = \lambda \int_0^1 b(t)m(s)x(s, \lambda)\, ds + f(t), \;\; t \in [0, 1].$$

Let us have irregular case of $a(0) = 1$. Let $b(0) \neq 0$, i.e. condition **I** is not fulfilled,

$$\frac{d}{d\lambda} A(\lambda)\big|_{\lambda=0} = b(0) \int_0^1 m(s)a(s)\, ds.$$

Let

$$\int_0^1 m(s)a(s)\, ds \neq 0.$$

Then all the conditions of Theorem 3 are fulfilled for $p = 1$. Then equation has solution $x(\lambda)$ for $|\lambda| > 0$ with 1st order pole at point $\lambda = 0$. The desired solution is the following

$$x(t, \lambda) = f(t, \lambda) - \frac{b(t)}{b(0)} f(0) + a(t)x(0, \lambda),$$

where the load $x(0, \lambda)$ is constructed as follows

$$x(0, \lambda) \equiv \frac{1}{(a, m)} \left[-\frac{f(0)}{\lambda b(0)} - (f, m) + \frac{f(0)}{b(0)}(b, m) \right],$$

where $(a, m) = \int_0^1 a(t)m(t)dt$, $(f, m) = \int_0^1 f(t)m(t)dt$, $(b, m) = \int_0^1 b(t)m(t)dt$. In this example the solution is constructed in an explicit form.

4 Conclusion and generalizations

The linear Fredholm integral functional equations of the second kind with linear functionals are studied. Necessary and sufficient conditions are formulated. Constructive methods are proposed for both regular and irregular cases. The solution in form of a Taylor series is constructed in terms of powers of the parameters. In the irregular case, the solution is constructed as a Laurent series of powers of the parameters. The constructive theory and methods are demonstrated using a model example. The case of $A_0 \neq E$ remained not addressed in this paper. The most complete results can be derived for the case of symmetric matrix A_0. In that case solution of equation (1) can be also presented as Laurent series with pole at point $\lambda = 0$. The corresponding sufficient condition can be derived based on generalized Jordan chains of the theory of perturbed nonlinear operators [7]. The bifurcation theory of nonlinear loaded integral equations, using the approach of this article in combination with representation theory and group symmetry [19], will also be addressed in future works. Some results in this direction are published in [4, 5, 14]. The numerical solution of Fredholm integral-functional equations of the second kind with linear functionals and parameter will be also addressed in future works.

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