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On recursive 1-differentiability of the quasigroup prolongations

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Abstract. The recursive differentiability of finite binary quasigroups is investigated. We consider the Bruck and Belousov constructions of prolongation of finite quasigroups and give necessary and sufficient conditions when such prolongations are recursively 1-differentiable.

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The first construction of a quasigroup prolongation was proposed by Bruck (see [1]) for the case of idempotent quasigroups in 1944. However, the notion of prolongation was introduced by Belousov (see [2]) in 1967. Constructions of prolongations of finite quasigroups have been given by Osborn (1961), Yamamoto (1961), Denes and Pasztor (1963), Belousov and Belyavskaya (1968), Belyavskaya (1969), Deriyenko and Dudek (2008, 2013) and others (see [7]).

Belousov considered a construction of prolongations based on complete mappings [2]. Recall that a complete mapping of a quasigroup $Q, \cdot)$ is a bijection $x \mapsto \theta(x)$ of Q onto Q such that $x \cdot \theta(x) = \theta_1(x)$ is also a bijective mapping of Q onto Q. The determination of all quasigroups, in particular groups, which possess a complete mapping remains at present an open problem [7]. In finite case, the complete mappings of quasigroups define transversals of their Cayley tables. A transversal of a latin square of order q is a set of q cells, taken by one from each row and each column, such that the elements in these cells are pairwise different.

Let $(Q, \cdot)$ be a finite quasigroup of order q, and let $\sigma : Q \mapsto Q$ be a complete mapping. Then $\{(x, \sigma(x)) | x \in Q\}$ is a transversal of the latin square given by the Cayley table of $(Q, \cdot)$. The prolongation $(Q', \circ)$ of $(Q, \cdot)$, where $Q' = Q \cup \{\xi\}$ and $\xi \notin Q$, considered by Belousov, is defined as follows:

$$x \circ y = \begin{cases} x \cdot y \text{ if } y \neq \sigma(x) \text{ and } x,\ y \in Q; \\ \xi \text{ if } y = \sigma(x) \text{ and } x,\ y \in Q; \\ x \cdot \sigma(x) \text{ if } y = \xi \text{ and } x \in Q; \\ \sigma^{-1}(y) \cdot y \text{ if } x = \xi \text{ and } y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases}$$

Analogously, we may construct prolongations of order $q + k$ if $(Q, \cdot)$ has k pairwise distinct transversals.

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In the present work we study the recursive differentiability of Bruck and Belousov prolongations, obtained by adding one element to a finite quasigroup.

The notions of recursive derivative and recursively r-differentiable k-quasigroup ($r \geq 0, k \geq 2$) have been introduced in [3] in connection with complete k-recursive codes.

Let Q be a finite set of q elements. Any nonempty subset C of Q^n is called an n-code (or a code of length n) over the alphabet Q. An n-code $C \subseteq Q^n$, where $|Q| = q$, with the minimum Hamming distance d, is called an $[n, k, d]_q$-code if $|C| = q^k$. It is known that the parameters of an $[n, k, d]_q$-code satisfy the inequality $d \leq n - k + 1$ [7]. An $[n, k, d]_q$-code with $d = n - k + 1$, i.e. which attains the Singleton bound, is called an MDS-code. At present it is an open problem to determine all values of the parameters q, n and d (for a fixed $k \geq 2$) such that there exist $[n, k, d]_q$-codes meeting the Singleton bound.

A code C of length n over an alphabet Q is called a complete k-recursive code, where $1 \leq k \leq n$, if there exists a mapping $f : Q^k \mapsto Q$ such that the components of every code word $u = (u_0, u_1, ..., u_{n-1}) \in C$ satisfy the conditions:

$$u_{i+k} = f(u_i, u_{i+1}, ..., u_{i+k-1}),$$

for every $i = 0, 1, ..., n - k$. So, if C is a complete k-recursive code of length n, over an alphabet Q, then there exist the mappings $f^{(0)}, f^{(1)}, ..., f^{(n-k-1)} : Q^k \mapsto Q$ such that $C = \{(x_1, ..., x_k, f^{(0)}(x_1^k), ..., f^{(n-k-1)}(x_1^k)) \mid x_1, ..., x_k \in Q\}$, where

$f^{(0)}(x_1^k) = f(x_1^k)$,
$f^{(1)}(x_1^k) = f(x_2, ..., x_k, f^{(0)}(x_1^k))$,
....
$f^{(t)}(x_1^k) = f(x_{t+1}, ..., x_k, f^{(0)}(x_1^k), ..., f^{(t-1)}(x_1^k))$, for $t < k$, and
$f^{(t)}(x_1^k) = f(f^{(t-k)}(x_1^k), ..., f^{(t-1)}(x_1^k))$, for $t \geq k$.

The mapping $f^{(t)}(x_1^k)$, where $t \geq 0$, is called the recursive derivative of order t of f. We say that a k-ary quasigroup (Q, f) is recursively s-differentiable if its recursive derivatives $f^{(1)}, ..., f^{(s)}$ are quasigroup operations. A complete k-recursive code $C = \{(x_1, ..., x_k, f^{(0)}(x_1^k), ..., f^{(n-k-1)}(x_1^k)) \mid x_1, ..., x_k \in Q\}$ is an MDS-code if and only if the system of k-recursive derivatives $\{f^{(0)}, ..., f^{(n-k-1)}\}$ is strongly orthogonal [3, 6]. As a corollary from this result we get that if the given above code C attains the Singleton bound then the k-ary operation f is recursively $(n-k-1)$-differentiable.

As orthogonal systems of binary quasigroups are strongly orthogonal, we obtain the following statement.

Theorem 1 [3] *A complete 2-recursive code of length n*

$$C = \{(x, y, f^{(0)}(x, y), ..., f^{(n-3)}(x, y)) \mid x, y \in Q\}$$

attains the Singleton bound if and only if (Q, f) is a recursively $(n-3)$-differentiable quasigroup. In this case, $\{f^{(0)}, ..., f^{(n-3)}\}$ is an orthogonal system of quasigroups.

It follows from Theorem 1 that:

1) a binary finite quasigroup (Q, f) is recursively r-differentiable if and only if the complete 2-recursive code

$$C = \{(x, y, f^{(0)}(x, y), ..., f^{(r)}(x, y)) \mid x, y \in Q\}$$

is an MDS-code;
2) the maximum order r of recursive differentiability of a finite binary quasigroup of order q satisfies the inequality $r \leq q - 2$ (see [5]).

Various methods of construction of binary recursively differentiable quasigroups are given in [3-6]. In particular, it is proved in [3] that, for every positive integer q, excepting 1, 2, 6, and possibly 14, 18, 24 and 42, there exist recursively 1-differentiable binary quasigroups of order q. Later, in 2009, it was shown that there exist recursively 1-differentiable quasigroups of order 42 (see [4]), but the question is still opened for 14, 18 and 24.

Another open problem is to determine the maximum order r of the recursive differentiability of a finite k-quasigroup. As it was mentioned above, in the binary case we have $r \leq q - 2$ and there exist recursively $(q - 2)$-differentiable binary quasigroups of every primary order $q \geq 3$ [3]. Necessary and sufficient conditions when a binary finite abelian group is recursively r-differentiable, for $r \geq 1$, are given in [6]. A generalization of this result for a class of n-ary groups is considered in [5]. Also a table with maximum known values of r for binary finite quasigroups of order up to 200 is given in [5], where it is shown, in particular, that there exist finite recursively 1-differentiable n-quasigroups of every odd order $q \geq 3$, for every $n \geq 2$.

Our aim in the present paper is to find necessary and sufficient conditions when the prolongations of finite binary quasigroups, obtained using Bruck and Belousov constructions, are recursively 1-differentiable. Let $(Q, \cdot)$ be a finite quasigroup of order n and $Q = \{1, 2, ..., n\}$ such that the mapping $x \mapsto x \cdot x$ is a bijection. Then the main diagonal of the Cayley table of $(Q, \cdot)$ is a transversal, which entries are given by the mapping $\theta : Q \mapsto Q, \theta(x) = x \cdot x$. As it was mentioned above, Bruck considered such prolongations for idempotent quasigroups, i.e. in the case $\theta = \epsilon$ be the identical mapping on Q.

Following Bruck's idea, the operation of the prolongation $(Q', \circ)$ of a quasigroup $(Q, \cdot)$, where $Q = \{1, ..., n\}$ and $Q' = Q \cup \{\xi\}, \xi \notin Q$, is defined as follows:

$$x \circ y = \begin{cases} x \cdot y \text{ if } x \neq y \text{ and } x,\ y \in Q; \\ \xi \text{ if } x = y \text{ and } x \in Q; \\ \theta(x) \text{ if } y = \xi \text{ and } x \in Q; \\ \theta(y) \text{ if } x = \xi \text{ and } y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases} \tag{1}$$

So, the prolongation $(Q', \circ)$ is a quasigroup with the Cayley table:

$\circ$	1	...	n	ξ
1	ξ	...	...	$\theta(1)$
...	...	...	...	...
n	...	...	ξ	$\theta(n)$
ξ	$\theta(1)$	...	$\theta(n)$	ξ

Table 1

where $x \circ y = x \cdot y$, for every $x \neq y$ from Q.

Remark that not every transversal on the main diagonal gives 1-differentiable prolongations as it is shown in the following statement.

Proposition 1. *Let $(Q, \cdot)$ be a finite quasigroup such that the mapping $\theta : Q \mapsto Q$, $\theta(x) = x \cdot x$ is a bijection. If the prolongation $(Q', \circ)$, given by (1), where $Q' = Q \cup \{\xi\}$, $\xi \notin Q$, is a quasigroup, then $\theta(x) \neq x, \forall x \in Q$.*

Proof. Indeed, if there exists an element $a \in Q$ such that $a = \theta(a) = a \cdot a$, then using (1) we get: $a \overset{1}{\circ} a = a \cdot (a \cdot a) = a \cdot a = a$ and $\xi \overset{1}{\circ} a = a \cdot (\xi \cdot a) = a \cdot \theta(a) = a \cdot a = a$, so $(Q', \circ)$ can not be a quasigroup. □

Lemma 1. *Let $(Q, \cdot)$ be a finite quasigroup of order n, $Q = \{1, ..., n\}$ and $Q' = Q \cup \{\xi\}$ where $\xi \notin Q$. If the mapping $\theta : Q \mapsto Q$, $\theta(x) = x \cdot x$ is a bijection and $\theta(x) \neq x$, $\forall x \in Q$, then the recursive derivative of order 1 of the operation " $\circ$ ", given in (1), is the following:*

$$x \overset{1}{\circ} y = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq x \cdot y, \ x \neq y, \ x, y \in Q; \\ \xi \text{ if } y = x \cdot y \ x \neq y, \ x, y \in Q; \\ \theta(y) \text{ if } x = y, \ x \in Q; \\ \theta^2(x) \text{ if } y = \xi, \ x \in Q; \\ y \cdot \theta(y) \text{ if } x = \xi, \ y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases} \tag{2}$$

Proof. Using (1) and the fact that $x \overset{1}{\circ} y = y \circ (x \circ y), \forall x, y \in Q'$, we have:

$$x \overset{1}{\circ} y = \begin{cases} y \circ (x \cdot y) \text{ if } x \neq y \text{ and } x, y \in Q; \\ y \circ \xi \text{ if } x = y, \ y \in Q; \\ \xi \circ \theta(x) \text{ if } y = \xi, \ x \in Q; \\ y \circ \theta(y) \text{ if } x = \xi, \ y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases}$$

Now, using (1) for " $\circ$ " in the previous formulas, we get:

$$x \overset{1}{\circ} y = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq x \cdot y, \ x \neq y \text{ and } x, y \in Q; \\ \xi \text{ if } y = x \cdot y, \ x \neq y \text{ and } x, y \in Q; \\ \theta(y) \text{ if } x = y, \ y \in Q; \\ \theta^2(x) \text{ if } y = \xi, \ x \in Q; \\ y \cdot \theta(y) \text{ if } x = \xi, \ y \neq \theta(y), \ y \in Q; \\ \xi \text{ if } x = \xi, \ y = \theta(y), \ y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases}$$

If the mapping $\theta : Q \mapsto Q, \ \theta(x) = x \cdot x$ is a bijection and $\theta(x) \neq x, \ \forall x \in Q$, then the prolongation $(Q', \circ)$ is a quasigroup and its recursive derivative $(\overset{1}{\circ})$ is defined as it is shown in (2). □

Remark 1. According to Lemma 1, the Cayley table of the recursive derivative $(Q', \overset{1}{\circ})$ is the following:

$\overset{1}{\circ}$	...	x	...	y	...	ξ
...	...	...	...	...	...	...
x	...	$\theta(x)$	...	z	...	$\theta^2(x)$
...	...	...	...	...	...	...
ξ	...	$x \cdot \theta(x)$	...	...	...	ξ

Table 2

where

$$z = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq x \cdot y; \\ \xi \text{ if } y = x \cdot y. \end{cases}$$

Theorem 2. *Let $(Q, \cdot)$ be a finite quasigroup such that the mapping $\theta : Q \mapsto Q$, $\theta(x) = x \cdot x$ is a bijection and $\theta(x) \neq x, \forall x \in Q$. Then the prolongation $(Q', \circ)$ obtained using Bruck's construction, where $Q' = Q \cup \{\xi\}$, $\xi \notin Q$, is recursively 1-differentiable if and only if the following conditions are satisfied:*
1. $\{f_x \mid x \in Q\} = Q$, where $f_x \cdot x = x, \forall x \in Q$;
2. θ is a complete mapping of $(Q, \cdot)$;
3. for each $x \in Q, \{\theta(x), y \cdot (x \cdot y), \theta^2(x) \mid y \in Q,\ x \neq y,\ y \neq x \cdot y\} = Q$.

Proof. According to Proposition 1, the condition $\theta(x) \neq x, \forall x \in Q$, implies the fact that the prolongation $(Q', \circ)$ is a quasigroup, so the equation $x \overset{1}{\circ} a = b \Leftrightarrow a \circ (x \circ a) = b$ has a unique solution in $(Q', \overset{1}{\circ})$ and consequently, the rows in Table 2 are permutations of Q'. For $x, y \in Q$, the entry of the cell (x, y) is ξ if and only if $y = x \cdot y$, i.e. if and only if $x = f_y$ is the left local unit of y. Thus ξ will appear exactly once in each row and each column of Table 2 if and only if $\{f_y \mid y \in Q\} = Q$. The row of the element ξ in Table 2 is a permutation of Q' if and only if $x \mapsto x \cdot \theta(x)$ is a bijection on Q, i.e. if and only if θ is a complete mapping of $(Q, \cdot)$.

Finally, the row of $x \in Q$ is a permutation of Q' if and only if

$$\{\theta(x), y \cdot (x \cdot y), \theta^2(x) \mid x \neq y, y \neq x \cdot y, y \in Q\} = Q.$$

□

Example 1. The prolongation of the quasigroup $(Q, \cdot)$, obtained using the transversal T={(1,1),(2,2),(3,3)}, is recursively 1-differentiable.

$\cdot$	1	2	3
1	**2**	1	3
2	1	**3**	2
3	3	2	**1**

$\circ$	1	2	3	ξ
1	ξ	1	3	2
2	1	ξ	2	3
3	3	2	ξ	1
ξ	2	3	1	ξ

$\overset{1}{\circ}$	1	2	3	ξ
1	2	1	ξ	3
2	ξ	3	2	1
3	3	ξ	1	2
ξ	1	2	3	ξ

As it was mentioned above, the Belousov's idea of prolongation uses an arbitrary transversal of the Cayley table, not necessarily one on the main diagonal.

Let $\{(x, \theta(x)) \mid x \in Q\}$, where $\theta \in S_Q$, be a transversal of a finite quasigroup $(Q, \cdot)$. Then the mapping $\theta' : Q \to Q, \theta'(x) = x \cdot \theta(x)$ is a bijection. Following the Bruck's idea, Belousov considered the prolongation $(Q', \circ)$, where $Q' = Q \cup \{\xi\}, \xi \notin Q$ and

$$x \circ y = \begin{cases} x \cdot y \text{ if } y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \xi \text{ if } y = \theta(x) \text{ and } x,\ y \in Q; \\ \theta'(\theta^{-1}(y)) \text{ if } x = \xi \text{ and } y \in Q; \\ \theta'(x) \text{ if } y = \xi \text{ and } x \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases} \tag{3}$$

Remark 2. If θ' is a bijection then $(Q', \circ)$ is a quasigroup with the following Cayley table:

$\circ$	...	$\theta(x)$	...	y	...	ξ
...	...	...	...	...	...	...
x	...	$\theta'(x)$	...	$x \cdot y$	...	$\theta'(x)$
...	...	...	...	...	...	...
ξ	...	...	...	$\theta'(\theta^{-1}(y))$	...	ξ

Table 3

Let $(Q, \cdot)$ be a finite quasigroup and $\theta \in S_Q$ such that $\theta' : Q \to Q, \theta'(x) = x \cdot \theta(x)$ is a bijection. Then the recursive derivative $(Q', \overset{1}{\circ})$ of the prolongation $(Q', \circ)$ given in (3) is the following:

$$x \overset{1}{\circ} y = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq \theta(x \cdot y),\ y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \xi \text{ if } y = \theta(x \cdot y),\ y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \theta'(\theta(x)) \text{ if } y = \theta(x) \text{ and } x, y \in Q; \\ y \cdot \theta'(\theta^{-1}(y)) \text{ if } y \neq \theta(\theta'(\theta^{-1}(y))),\ x = \xi \text{ and } y \in Q; \\ \xi \text{ if } y = \theta(\theta'(\theta^{-1}(y))),\ x = \xi \text{ and } y \in Q; \\ \theta'(\theta^{-1}(\theta'(x))) \text{ if } y = \xi \text{ and } x \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases} \tag{4}$$

Proof. Indeed, (4) follows from (3), using the definition of the recursive derivative $x \overset{1}{\circ} y = y \circ (x \circ y), \forall x, y \in Q$. $\square$

Remark 3. If $y = \theta(\theta'(\theta^{-1}(y)))$, where $y \in Q$, then $\xi \overset{1}{\circ} y = \xi = \xi \overset{1}{\circ} \xi$, so $(Q', \overset{1}{\circ})$ is not a quasigroup.

Now, using (4) and Remark 3, we get the following statement.

Lemma 2. *Let $(Q, \cdot)$ be a finite quasigroup, $\theta \in S_Q$ such that $\theta' : Q \mapsto Q$, $\theta'(x) = x \cdot \theta(x)$ is a bijection and $y \neq \theta(\theta'(\theta^{-1}(y))), \forall y \in Q$. Then the recursive derivative*

$(Q', \overset{1}{\circ})$ *of the Belousov's prolongation* $(Q', \circ)$ *is:*

$$x \overset{1}{\circ} y = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq \theta(x \cdot y),\ y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \xi \text{ if } y = \theta(x \cdot y),\ y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \theta'(\theta(x)) \text{ if } y = \theta(x) \text{ and } x, y \in Q; \\ y \cdot \theta'(\theta^{-1}(y)) \text{ if } x = \xi \text{ and } y \in Q; \\ \theta'(\theta^{-1}(\theta'(x))) \text{ if } y = \xi \text{ and } x \in Q; \\ \xi \textit{ if } x = y = \xi. \end{cases} \tag{5}$$

Proof. The proof follows from (4) and the condition $y \neq \theta(\theta'(\theta^{-1}(y))), \forall y \in Q$. $\square$

Remark 4. The Cayley table of $(Q', \overset{1}{\circ})$, given in (5) is the following:

$\overset{1}{\circ}$	...	$\theta(x)$	...	y	...	ξ
...	...	...	...	...	...	...
x	...	$\theta'(\theta(x))$	...	w	...	$\theta'(\theta^{-1}(\theta'(x))$
...	...	...	...	...	...	...
ξ	...	...	...	$y \cdot \theta'(\theta^{-1}(y))$	...	ξ

Table 4

where

$$w = \begin{cases} y \cdot xy \text{ if } y \neq \theta(x \cdot y),\ y \neq \theta(x); \\ \xi \text{ if } y = \theta(x \cdot y),\ y \neq \theta(x). \end{cases}$$

Theorem 3. *Let* $(Q, \cdot)$ *be a finite quasigroup,* $\theta \in S_Q$ *such that the mapping* $\theta' : Q \mapsto Q$, $\theta'(x) = x \cdot \theta(x)$ *is a bijection and* $\theta^{-1}(y) \neq \theta'(\theta^{-1}(y)), \forall y \in Q$. *Then the Belousov's prolongation* $(Q', \circ)$ *is recursively 1-differentiable if and only if the following conditions hold:*
1. $\{\theta^{-1}(y)/y \mid y \in Q\} = Q$*;*
2. the mapping $y \mapsto y \cdot \theta'(\theta^{-1}(y))$ *is a bijection on* Q*;*
3. for each $x \in Q$, $\{\theta'(\theta(x)), y \cdot xy, \theta'(\theta^{-1}(\theta'(x)))| y \neq \theta(x \cdot y), y \neq \theta(x), y \in Q\} = Q$.

Proof. According to Belousov's construction, $(Q', \circ)$ is a quasigroup, so the equation $x \overset{1}{\circ} a = b \Leftrightarrow a \circ (x \circ a) = b$ has a unique solution in Q', for every $a, b \in Q'$. Thus the rows in the Cayley table (5) are permutations of Q'. The element ξ appears in a cell (x, y) with $x, y \in Q$ if $y = \theta(x \cdot y),\ y \neq \theta(x)$, i.e. if $x = \theta^{-1}(y)/y$. If $(Q', \overset{1}{\circ})$ is a quasigroup, then $\{\theta^{-1}(y)/y \mid y \in Q\} = Q$.

According to Table 4, the row of ξ is a permutation of Q' if and only if the mapping $y \mapsto y \cdot \theta'(\theta^{-1}(y))$ is a bijection on Q.

Finally, the row of $x \in Q$ in Table 4, is a permutation of Q' if and only if the third condition is fulfilled. $\square$

Example 2. The prolongation of the quasigroup $(Q, \cdot)$, obtained using the transversal T={(1,2),(2,1),(3,3)}, is recursively 1-differentiable.

$\cdot$	1	2	3
1	2	**3**	1
2	**1**	2	3
3	3	1	**2**

$\circ$	1	2	3	ξ
1	2	ξ	1	3
2	ξ	2	3	1
3	3	1	ξ	2
ξ	1	3	2	ξ

$\overset{1}{\circ}$	1	2	3	ξ
1	ξ	1	3	2
2	3	2	ξ	1
3	1	ξ	2	3
ξ	2	3	1	ξ

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