BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(101), 2023, Pages 8–15
ISSN 1024–7696, E-ISSN 2587–4322

Some families of quadratic systems with at most one limit cycle

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Abstract. The work of Chicone and Shafer published in 1982 together with the work of Bamon published in 1986 proved that any polynomial differential system of degree two has finitely many limit cycles. But the problem remains open of providing a uniform upper bound for the maximum number of limit cycles that a polynomial differential system of degree two can have, i.e. the second part of the 16th Hilbert problem restricted to the polynomial differential systems of degree two remains open. Here we present six subclasses of polynomial differential systems of degree two for which we can prove that an upper bound for their maximum number of limit cycles is one.

Mathematics subject classification: 34C05.
Keywords and phrases: quadratic systems, 16th Hilbert problem, limit cycles.

1 Introduction and statement of the main results

We deal with polynomial differential systems in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y).$$

The degree of such a polynomial system is the maximum of the degrees of the polynomials P and Q. In what follows the polynomial differential systems of degree 2 are simply called *quadratic systems*.

We recall that a *limit cycle* of a differential system is a periodic orbit of this system isolated in the set of all periodic orbits of the system. As far as we know the notion of limit cycle appeared in the work of Poincaré [14] in the year 1885.

At the Second International Congress of Mathematicians, held in Paris in 1900, Hilbert [8] proposed his famous 16th problem, whose second part essentially says: *Find an upper bound for the maximum number of limit cycles that the polynomial differential systems in $\mathbb{R}^2$ of a given degree can have.*

The works of Chicone and Shafer [5] and of Bamon [1] proved that any polynomial differential system of degree 2 has finitely many limit cycles. This result uses previous results of Ilyashenko [9]. Up to now the second part of the 16th Hilbert problem remains unsolved, also for the quadratic systems.

In 1957 Petrovskii and Landis [12] claimed that the polynomial differential systems of degree $n = 2$ have at most 3 limit cycles. Soon (in 1959) a gap was found

DOI: https://doi.org/10.56415/basm.y2023.i1.p8

in the arguments of Petrovskii and Landis, see [13]. Later, Lan Sun Chen and Ming Shu Wang [3] in 1979, and Songling Shi [16] in 1982, provided the first quadratic systems having 4 limit cycles, and up to now 4 is the maximum number of limit cycles known for a quadratic system.

We recall the following three well known properties of quadratic systems.

(a) In the region limited by a periodic orbit of a quadratic system there is a unique equilibrium point, see Theorem 2 of Coppel [6], or Theorem 2.8 of Chicone and Jinghuang [4].

(b) A periodic orbit of a quadratic system surrounds a focus or a center, proved by Vorob'ev [17], see also Theorem 6 of Coppel [6].

(d) Quadratic systems having a center have no limit cycles, see Vulpe [18] and Schlomiuk [15].

From these three properties if follows that if a quadratic system has a limit cycle this must surround a focus.

Let O be a focus or a center of a quadratic system, without loss of generality we can assume that O is localized at the origin of coordinates, otherwise we do a translation sending O to the origin of coordinates. Kaptein [10, 11] proved that any quadratic system having a focus or a center at the origin of coordinates can be written as (see also Bautin [2])

$$\begin{aligned} \dot{x} &= \lambda_1 x - y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{aligned} \tag{1}$$

In order to avoid subindexes we denote

$$\lambda_1 = \lambda, \quad \lambda_2 = a, \quad \lambda_3 = b, \quad \lambda_4 = c, \quad \lambda_5 = d, \quad \lambda_6 = e.$$

Then system (1) becomes

$$\begin{aligned} \dot{x} &= \lambda x - y - bx^2 + (2a + d)xy + ey^2, \\ \dot{y} &= x + \lambda y + ax^2 + (2b + c)xy - ay^2. \end{aligned} \tag{2}$$

The goal of this paper is to give conditions on the parameters of system (2) for the presence of a maximum of one limit cycle for the system surrounding the origin. For this we rely on the paper [7] where a theorem is stated giving conditions for having at most three limit cycles in an Abel differential equation.

A good tool for studying the possible limit cycles surrounding the origin O of the quadratic system (2) is to write this quadratic system in polar coordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$. Then system (2) becomes

$$\begin{aligned} \dot{r} &= \lambda r + f(\theta) r^2, \\ \dot{\theta} &= 1 + g(\theta) r, \end{aligned} \tag{3}$$

where

$$
\begin{aligned}
f(\theta) &= -a\sin^3\theta + (2b+c+3)\sin^2\theta\cos\theta + (3a+d)\sin\theta\cos^2\theta - b\cos^3\theta,\\
g(\theta) &= -3\sin^3\theta - (3a+d)\sin^2\theta\cos\theta + (3b+c)\sin\theta\cos^2\theta + a\cos^3\theta.
\end{aligned}
\tag{4}
$$

Note that $f(\theta)$ and $g(\theta)$ are homogeneous trigonometric polynomials of degree three.

We define the polynomials

$$
\begin{aligned}
F(z) &= -az^3 + (2b+c+3)z^2 + (3a+d)z - b,\\
G(z) &= -3z^3 - (3a+d)z^2 + (3b+c)z + a,
\end{aligned}
$$

note that $f(\theta) = \cos^3\theta F(\tan\theta)$ and $g(\theta) = \cos^3\theta G(\tan\theta)$.

Here first we classify all quadratic systems whose polynomial $G(z)(\lambda G(z) - F(z))$ satisfies the following two properties:

(P1) it has degree six, and

(P2) for all $z \in \mathbb{R}$ the value of $G(z)(\lambda G(z) - F(z))$ is either ≥ 0, or $= 0$, or ≤ 0.

Theorem 1. *Every quadratic system* (2) *satisfying properties* ($P1$) *and* ($P2$) *must be one of the following six forms of quadratic systems*

$$
\begin{aligned}
\dot{x} &= \lambda x - y - bx^2 - 2axy - a^2y^2/b,\\
\dot{y} &= x + \lambda y + ax^2 + (a^2 - b^2)xy/b - ay^2
\end{aligned}
\tag{5}
$$

(i.e. $d = 0$, $c = (a^2 - 3b^2)/b$ *and* $e = -a^2/b$ *in* (2)*), with* $b \neq 0$*;*

$$
\begin{aligned}
\dot{x} &= \lambda x - y - cy^2,\\
\dot{y} &= x + \lambda y + cxy
\end{aligned}
\tag{6}
$$

(i.e. $a = b = d = 0$ *and* $e = c$ *in* (2)*), with* $c \neq 0$*;*

$$
\begin{aligned}
\dot{x} &= \lambda x - y + dxy + ey^2,\\
\dot{y} &= x + \lambda y + cxy
\end{aligned}
\tag{7}
$$

(i.e. $a = b = 0$ *in* (2)*), with* $c^2 + d^2 + e^2 \neq 0$ *and* $\Delta_i(\lambda, 0, 0, c, d, e) > 0$ *for* $i = 1, 2$*;*

$$
\begin{aligned}
\dot{x} &= \lambda x - y + (2a+d)xy + ey^2,\\
\dot{y} &= x + \lambda y + ax^2 + cxy - ay^2
\end{aligned}
\tag{8}
$$

(i.e. $b = 0$ *in* (2)*), where* $\Delta_i(\lambda, a, 0, c, d, e) > 0$ *for* $i = 1, 2$*,*
$c = -a(2a+d-e)(2a+d+e)/((2a+d)e)$ *and* $(2a+d)e \neq 0$*;*

$$
\begin{aligned}
\dot{x} &= \lambda x - y - bx^2 + (2a+d)xy + ey^2,\\
\dot{y} &= x + \lambda y + ax^2 + cxy - ay^2
\end{aligned}
\tag{9}
$$

(i.e. $2b + c = c$ *in* (2)*), where* $\Delta_i(\lambda, a, b, c, d, e) > 0$ *for* $i = 1, 2$*, and* $c = -a\left(2ab + 2ae + bd + de + (b-e)\sqrt{(2a+d)^2 + 4be}\right)/(2be)$ *and* $be \neq 0$*;*

$$
\begin{aligned}
\dot{x} &= \lambda x - y - bx^2 + (2a+d)xy + ey^2,\\
\dot{y} &= x + \lambda y + ax^2 + cxy - ay^2
\end{aligned}
\tag{10}
$$

(i.e. $2b + c = c$ *in* (2)*), where* $\Delta_i(\lambda, a, b, c, d, e) > 0$ *for* $i = 1, 2$*, and* $c = -a\left(2ab + 2ae + bd + de - (b-e)\sqrt{(2a+d)^2 + 4be}\right)/(2be)$ *and* $be \neq 0$*.*
The functions Δ_i *for* $i = 1, 2$ *are defined in Section* 2*.*

Theorem 2. *The six quadratic families of systems of Theorem* 1 *have only one equilibrium point, the origin of coordinates. Moreover all these quadratic systems have at most one limit cycle, and when it exists it surrounds the origen of coordinates.*

Theorems 1 and 2 are proved in Section 2.

2 Proof of Theorems 1 and 2

Statement (a) of the next proposition is proved in statement (a) of Proposition 8 of Gasull and Llibre [7], and statement (b) of the next proposition is proved in statement (b) of Theorem C also in [7].

Proposition 1. *Let* $A(\theta) = g(\lambda g - f)$*, where the functions* $f(\theta)$ *and* $g(\theta)$ *are defined in* (4)*. Then the following statements hold.*

(a) If $A(\theta) \neq 0$ *and either* $A(\theta) \geq 0$ *or* $A(\theta) \leq 0$*, then system* (2) *has at most one limit cycle surrounding the origin. Furthermore, it can exist only if* $\lambda sign(A(\theta)) < 0$.

(b) If $A(\theta) = 0$*, then system* (2) *has at most one limit cycle surrounding the origin.*

From Proposition 1 the next result follows immediately .

Corollary 1. *If for all values of* $z \in \mathbb{R}$ *the polynomial* $G(z)(\lambda G(z) - F(z))$ *is either* ≥ 0*, or* $= 0$*, or* ≤ 0*, then the differential system* (2) *has at most one limit cycle surrounding the origin of coordinates.*

If the polynomial $G(z)(\lambda G(z) - F(z))$ is the zero polynomial, then by Corollary 1 there is at most one limit cycle of system (2) surrounding the origin. Later on we will show that the six quadratic families of systems of Theorem 1 have only a unique equilibrium point, the origin. So Theorems 1 and 2 will be proved when the polynomial $G(z)(\lambda G(z) - F(z))$ is the zero polynomial. So in what follows we assume that this polynomial is distinct from zero.

By assumption (P1) the polynomial $G(z)(\lambda G(z) - F(z))$ has degree six, therefore both polynomials $G(z)$ and $\lambda G(z) - F(z)$ are of degree three, so they have at least one real root. Then such a real root must be common to the polynomials $G(z)$ and $\lambda G(z) - F(z)$, otherwise the assumption (P2) would not hold. Hence the resultant of the polynomials $G(z)$ and $\lambda G(z) - F(z)$ must be zero, i.e.

$$\begin{aligned} R(G, \lambda G - F) = \quad & \left((4a+d)^2 + (3b+c+e)^2\right)\Big(ad(2b+c)(b+e) + be(2b+c)^2 + \\ & 4a^4 + 4a^3 d + a^2\left(3b^2 + 2b(c+3e) + 2ce + d^2 - e^2\right)\Big). \end{aligned}$$

Now we consider two cases.
Case 1: $(4a+d)^2 + (3b+c+e)^2 = 0$. Then $d = -4a$, $e = -3b - c$. Therefore the roots of the polynomial $G(z)$ are $\pm i$ and $-a/(3b+c)$, note that $3b + c \neq 0$ because

the polynomial $G(z)$ has degree 3 and consequently it must have three roots. The roots of the polynomial $\lambda G(z) - F(z)$ are $\pm i$ and $-(b + \lambda a)/(a + \lambda(3b + c))$, and $a + \lambda(3b + c) \neq 0$ because the polynomial $\lambda G(z) - F(z)$ has degree 3.

In order that the polynomial $G(z)(\lambda G(z) - F(z))$ verify that $g \geq 0$ or $g \leq 0$ for all $z \in \mathbb{R}$, we need that the real root of the polynomials $G(z)$ and $\lambda G(z) - F(z)$ coincide, i.e.

$$-a/(3b + c) = -(b + \lambda a)/(a + \lambda(3b + c)). \tag{11}$$

Then if $b \neq 0$ we have that $c = (a^2 - 3b^2)/b$ and

$$G(z)(\lambda G(z) - F(z)) = \frac{a(\lambda a + b)(z^2 + 1)^2(az + b)^2}{b^2}.$$

Since the function $G(z)(\lambda G(z) - F(z))$ satisfies the assumptions of Corollary 1, so system (2) satisfying $c = (a^2 - 3b^2)/b$ reduces to system (5) and has at most one limit cycle, this limic cycle surrounds the origin. Furthermore also this system has a unique equilibrium point, the origin, as it is easy to check.

If $b = 0$ then from (11) we get that $a = 0$, and consequently

$$G(z)(\lambda G(z) - F(z)) = \lambda c^2 z^2 (1 + z^2)^2.$$

Again the function $G(z)(\lambda G(z) - F(z))$ satisfies the assumptions of Corollary 1, so system (2) satisfying $b = a = 0$ reduces to system (6) and has at most one limit cycle, and this limic cycle surrounds the origin. Furthermore also this system has a unique equilibrium point, the origin, as it is easy to verify.

We remark that if we impose that the polynomials $G(z)$ and $\lambda G(z) - F(z)$ be one a multiple of the other, or equivalently that they have exactly the same three roots, then we get exactly the previous two quadratic systems (5) and (6). Hence in what follows we can assume that the polynomials $G(z)$ and $\lambda G(z) - F(z)$ have different roots. Then in order that the polynomial $G(z)(\lambda G(z) - F(z))$ can satisfy the assumption (P2) and since the polynomials $G(z)$ and $\lambda G(z) - F(z)$ are cubic polynomials by the assumption (P1), they must have in common a real root, and the other roots cannot be the same for both polynomials, otherwise we will obtain the quadratic systems (5) and (6).

In summary, we can restrict our attention to the polynomials $G(z)$ and $\lambda G(z) - F(z)$ having a common real root and the other two roots non-real because if one of these two polynomials has the three real roots, then it is not possible that the polynomial $G(z)(\lambda G(z) - F(z))$ satisfies the assumption (P2), i.e. the polynomial $G(z)(\lambda G(z) - F(z))$ would change the sign because not all the real roots of the polynomials $G(z)$ and $(\lambda G(z) - F(z))$ would coincide. This implies that the discriminats of the polynomials $G(z)$ and $\lambda G(z) - F(z)$ must be positive, see an easy proof of this fact in the cubic equation of Wikipedia.

The discriminants $\Delta_i = \Delta_i(\lambda, a, b, c, d, e)$ for $i = 1, 2$ of the polynomials $G(z)$ and $\lambda G(z) - F(z)$ are respectively
$\Delta_1 = 108a^4 + 81a^2b^2 + 54a^2bc + 9a^2c^2 + 108a^3d + 54ab^2d + 36abcd + 6ac^2d + 36a^2d^2 + 9b^2d^2 + 6bcd^2 + c^2d^2 + 4ad^3 + 162a^2be + 108b^3e + 54a^2ce + 108b^2ce + 36bc^2e + 4c^3e +$

$54abde + 18acde - 27a^2e^2$,
and
$\Delta_2 = 108a^4 + 117a^2b^2 + 32b^4 + 90a^2bc + 48b^3c + 9a^2c^2 + 24b^2c^2 + 4bc^3 + 108a^3d + 60ab^2d + 42abcd + 6ac^2d + 36a^2d^2 + 4b^2d^2 + 4bcd^2 + c^2d^2 + 4ad^3 + 90a^2be + 48b^3e + 18a^2ce + 48b^2ce + 12bc^2e + 42abde + 12acde + 4bd^2e + 2cd^2e + 9a^2e^2 + 24b^2e^2 + 12bce^2 + 6ade^2 + d^2e^2 + 4be^3 - 4ab^3\lambda + 6ab^2c\lambda - 2ac^3\lambda + 36a^2bd\lambda + 24b^3d\lambda + 16b^2cd\lambda - 2bc^2d\lambda - 2c^3d\lambda + 18abd^2\lambda + 6acd^2\lambda + 4bd^3\lambda + 2cd^3\lambda + 12ab^2e\lambda - 12abce\lambda - 36a^2de\lambda - 12b^2de\lambda - 14bcde\lambda - 4c^2de\lambda - 18ad^2e\lambda - 2d^3e\lambda - 12abe^2\lambda + 6ace^2\lambda - 12bde^2\lambda - 2cde^2\lambda + 4ae^3\lambda + 216a^4\lambda^2 + 198a^2b^2\lambda^2 + 36b^4\lambda^2 + 144a^2bc\lambda^2 + 60b^3c\lambda^2 + 18a^2c^2\lambda^2 + 37b^2c^2\lambda^2 + 10bc^3\lambda^2 + c^4\lambda^2 + 216a^3d\lambda^2 + 102ab^2d\lambda^2 + 54abcd\lambda^2 + 72a^2d^2\lambda^2 - 8bcd^2\lambda^2 - 4c^2d^2\lambda^2 + 12ad^3\lambda^2 + d^4\lambda^2 + 252a^2be\lambda^2 + 144b^3e\lambda^2 + 72a^2ce\lambda^2 + 132b^2ce\lambda^2 + 34bc^2e\lambda^2 + 2c^3e\lambda^2 + 120abde\lambda^2 + 54acde\lambda^2 + 18bd^2e\lambda^2 + 8cd^2e\lambda^2 - 18a^2e^2\lambda^2 + 36b^2e^2\lambda^2 + 24bce^2\lambda^2 + c^2e^2\lambda^2 - 6ade^2\lambda^2 + 18ab^2c\lambda^3 + 12abc^2\lambda^3 + 2ac^3\lambda^3 + 36a^2bd\lambda^3 + 36b^3d\lambda^3 + 42b^2cd\lambda^3 + 16bc^2d\lambda^3 + 2c^3d\lambda^3 + 6abd^2\lambda^3 - 6acd^2\lambda^3 - 2bd^3\lambda^3 - 2cd^3\lambda^3 - 36abce\lambda^3 - 12ac^2e\lambda^3 - 36a^2de\lambda^3 - 36b^2de\lambda^3 - 42bcde\lambda^3 - 10c^2de\lambda^3 - 6ad^2e\lambda^3 + 18ace^2\lambda^3 + 108a^4\lambda^4 + 81a^2b^2\lambda^4 + 54a^2bc\lambda^4 + 9a^2c^2\lambda^4 + 108a^3d\lambda^4 + 54ab^2d\lambda^4 + 36abcd\lambda^4 + 6ac^2d\lambda^4 + 36a^2d^2\lambda^4 + 9b^2d^2\lambda^4 + 6bcd^2\lambda^4 + c^2d^2\lambda^4 + 4ad^3\lambda^4 + 162a^2be\lambda^4 + 108b^3e\lambda^4 + 54a^2ce\lambda^4 + 108b^2ce\lambda^4 + 36bc^2e\lambda^4 + 4c^3e\lambda^4 + 54abde\lambda^4 + 18acde\lambda^4 - 27a^2e^2\lambda^4$.
We recall that when the discriminant of a cubic polynomial is positive, then such a polynomial has a unique real root.
Case 2:
$ad(2b+c)(b+e) + be(2b+c)^2 + 4a^3(a+d) + a^2\left(3b^2 + 2b(c+3e) + 2ce + d^2 - e^2\right) = 0$.
This equation has the following seven sets of solutions

(s1) $a = e = 0$;

(s2) $b = e = 0$ and $d = -2a$;

(s3) $c = -((2b^2d + 4a^2(a+d) + a(3b^2 + d^2))/(b(2a+d)))$ and $e = 0$;

(s4) $a = b = 0$;

(s5) $c = -((a(2a+d-e)(2a+d+e))/((2a+d)e))$ and $b = 0$;

(s6) $c = -((2(a^2+2b^2)e + 2a^2(b+e) + ad(b+e) + a(b-e)\sqrt{(2a+d)^2 + 4be})/(2be))$;

(s7) $c = -((2(a^2+2b^2)e + 2a^2(b+e) + ad(b+e) - a(b-e)\sqrt{(2a+d)^2 + 4be})/(2be))$.

The polynomial $G(z)(\lambda G(z) - F(z))$ has degree less than 6 for the solutions (s1), (s2) and (s3), so we do not consider these three solutions. While for the solutions from (s4) to (s7) this polynomial has degree 6.

Every one of the solutions from (s4) to (s7) implies that the polynomials $G(z)$ and $\lambda G(z) - F(z)$ have at least one root in common, if additionally we impose that the discriminants of these two polynomials are positive, then these polynomials have one real root in common and two distinct conjugate complex roots. Additionally we shall prove that the quadratic systems satisfying some solution (sk) for $k = 4, \dots, 7$ have a unique equilibrium, the focus localized at the origin of coordinates, therefore by Corollary 1 we obtain that these four families of quadratic systems satisfying some solution (sk) for $k = 4, \dots, 7$ with $\Delta_1 > 0$, $\Delta_2 > 0$, cannot have more than one limit cycle surrounding the origin. Hence Theorems 1 and 2 will be proved.

Now we prove that the quadratic systems satisfying (sk) for $k = 4, \dots, 7$ have a unique equilibrium. Indeed, since the polynomial $\lambda G(z) - F(z)$ has a unique real root and two complex ones, and this real root also is the unique real root of the polynomial $G(z)$, it follows from systems (3) that systems (2) has only one finite equilibrium point, the origin of coordinates. Indeed, the equilibrium points (r^*, θ^*) of system (3) with $r^* \neq 0$ must satisfy that $\lambda g(\theta^*) - f(\theta^*) = 0$ and $r^* = -1/g(\theta^*)$, but if $\lambda g(\theta^*) - f(\theta^*) = 0$ then $1/g(\theta^*) = \infty$. Hence the unique equilibrium point of system (3) is the one with $r = 0$, i.e. the origin of coordinates.

We note that the solutions (s4), (s5), (s6) and (s7) provide the quadratic systems (7), (8), (9) and (10), respectively.

Acknowledgements

This work is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, the H2020 European Research Council grant MSCA-RISE-2017-777911, the Generalitat de Catalunya grant 2021 SGR 00113, and by the Acadèmia de Ciències i Arts de Barcelona.

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Received June 25, 2022