Counting configurations of limit cycles and centers

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Abstract. We present several results on the determination of the number and distribution of limit cycles or centers for planar systems of differential equations. In most cases, the study of a recurrence is one of the key points of our approach. These results include the counting of the number of configurations of stabilities of nested limit cycles, the study of the number of different configurations of a given number of limit cycles, the proof of some quadratic lower bounds for Hilbert numbers and some questions about the number of centers for planar polynomial vector fields.

Mathematics subject classification: 34C05, 34C07, 37C27.
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Dedicated to the memory of Professor Constantin Sibirschi

1 Introduction

In the qualitative study of differential systems in the plane, there are several questions about topological configurations that naturally lead to enumeration and combinatorial problems which, sometimes, can be approached by using recurrences.

This paper is devoted to study such type of questions for smooth planar differential systems,

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \tag{1}$$

In some parts of the work, the functions P and Q defining these differential equations will also be assumed to be polynomials.

After giving some definitions in Section 2, in Section 3 we will consider the growth of the number of stability configurations of n nested limit cycles of (1) in terms of their stability. We will show that this number of configurations can be explicitly determined using expressions involving the Fibonacci numbers, $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Although it may not be necessary to highlight the importance and ubiquity of the Fibonacci sequence both in arithmetic and geometry problems, and in some applied questions, we point out a couple of examples: their use in graph theory ([33]) and their earliest appearance in Indian mathematics to count the number of sums of 1 and 2 (taking into account the order of the addends) that add up to n. For instance, if $n = 4$, there appear $F_{n+1} = F_{4+1} = 5$ possibilities:

$$1 + 1 + 1 + 1, \; 2 + 1 + 1, \; 1 + 2 + 1, \; 1 + 1 + 2, \; 2 + 2.$$

Our second result deals with planar systems having exactly n limit cycles. In Section 4 we study the number of different configurations that these limit cycles can exhibit regarding only its topological distribution. In Figure 1, we show a colored illustration of the 20 different configurations of 5 limit cycles.
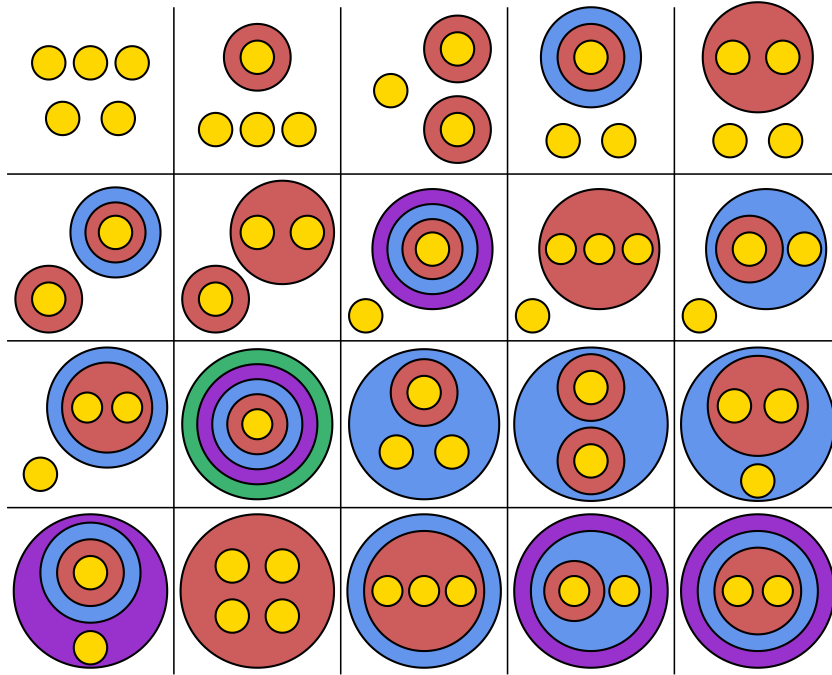

Figure 1. The 20 configurations in the case $n = 5$. The colors have a purely aesthetic function and mean nothing to the dynamics.

In our classification we do not take into account dynamical considerations such as the classification of the corresponding phase portraits. Notice that this point of view is totally different of that of Section 3 where the stability of the different periodic orbits is the key point to distinguish the different stability configurations. This problem can be seen as the pure geometrical problem of counting the number of ways of arranging n non-overlapping circles, which are in bijection with the different cases of unlabeled rooted trees with $n + 1$ nodes, [37]. The problem of finding this last number was studied by A. Cayley in his 1875's paper, [7], where a recurrence relation to compute this number is presented (see also [5, p. 43] or [15, p. 71]). In Proposition 3, we will give another recurrent expression. Our approach is self-contained and, to our knowledge, novel in the context of the study of limit cycles.

These two sections have no relation with existing literature where the number of possible phase portraits of some families of planar polynomial differential equations, modulo topological conjugacy (see for instance [3]) or modulo the existence of limit

cycles (see for instance [14]), is studied in detail.

Section 5 deals with Hilbert numbers for polynomial systems. Recall that when P and Q are arbitrary polynomials of degree at most n, the *Hilbert number* $\mathcal{H}(n)$ is the maximum number of limit cycles that these differential equations (1) can have, or infinity if there is no upper bound for the number of limit cycles. It is well-known that $\mathcal{H}(0) = \mathcal{H}(1) = 0$ and $\mathcal{H}(2) \geq 4$, but even for $n = 2$ it is not known whether $\mathcal{H}(2)$ is finite. Following [17], we present a very simple proof that the Hilbert numbers increase at least quadratically with n; the proof is based on the ideas of [9] and uses a very basic background on ordinary differential equations (ODEs) and recurrences.

Finally, in Section 6 we collect several known results about the maximum number of centers of planar polynomials systems of degree n, see also [16]. Special attention is paid to Hamiltonian and holomorphic centers, following [2, 10, 11, 35].

2   Some definitions

In the theory of planar differential systems, *a limit cycle* is a periodic orbit such that at least there is another trajectory that either spirals into it or from it as time approaches infinity. In this paper we will deal with analytic systems, hence a limit cycle is an isolated periodic orbit. Notice that even for $\mathcal{C}^\infty$-systems limit cycles can be non-isolated periodic orbits.

Consider an analytic planar differential system. Let γ be a limit cycle of this differential system. We will say that it is *stable*, and denote it as γ^s, if it is locally asymptotically stable. We say that it is *unstable*, and denote it by γ^u, if either it is a repelle or semistable (in this last case, either the nearby external paths converge to it and the nearby internal ones diverge from it, or the reciprocal situation is given).

We introduce some more preliminary definitions. Let c be a closed curve embedded in $\mathbb{R}^2$. A *configuration of cycles* is a finite set $\mathcal{C} = \{c_1, ..., c_n\}$ of disjoint simple closed curves. Following [29, 31] we say that two configurations of cycles $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if there exists a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(\mathcal{C}) = \mathcal{C}'$. Clearly the different configurations are characterized by the topology of the set $\mathbb{R}^2 \setminus \mathcal{C}$. It is also known that any configuration of cycles is homeomorphic to a set of disjoint circles, see again [29].

When the closed curves defining a configuration correspond to the limit cycles of a differential equation (1) we will say that it is a *limit cycles configuration*. If the stability of the limit cycles is also taken into account we will say that it is a *limit cycles stability configuration*.

If γ is a limit cycle, we will denote by C_γ the bounded open set with boundary γ. Consider now a differential equation (1) with finitely many periodic orbits (so, all them are limit cycles). We can introduce a natural partial order in the set $\mathcal{P}$ formed by the union of all limit cycles by defining $\gamma' \prec \gamma$ whenever $\gamma' \subset C_\gamma$ given $\gamma', \gamma \in \mathcal{P}$. With this order, $(\mathcal{P}, \prec)$ is a *partially ordered set (poset)* and the following concepts can be introduced:
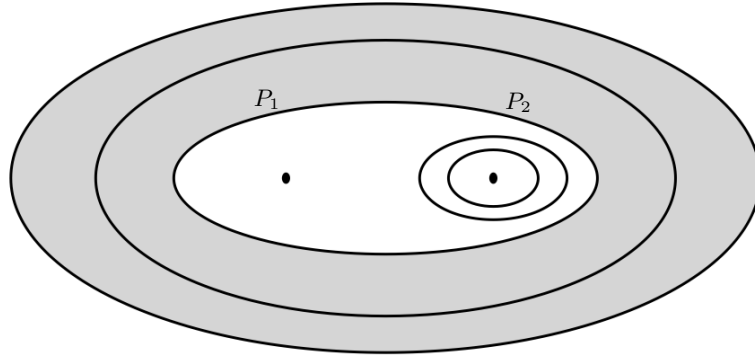
Figure 2. Two different nests of limit cycles. One of level 2, surrounding the singular point P_2, and another one, included in the shaded region, of level 3 surrounding the two singular points (and eventually others located in the white region). The level of the maximal limit cycle is 5.

- A maximal element γ of the poset $(\mathcal{P}, \prec)$ will be called *maximal limit cycle*. In this case, C_γ will be called the *domain* of γ and the number of cycles γ' such that $\gamma' \subset \overline{C_\gamma}$, the *level* of γ.

- We say that a subset $\mathcal{N} \subseteq \mathcal{P}$ is a *nest* if all periodic orbits $\gamma \in \mathcal{N}$ are such that all the corresponding C_γ sets contain the same singular points and moreover it is the biggest set in $\mathcal{P}$ with this property. The *level* of the nest $\mathcal{N}$ is its number of elements.

In Figure 2, we show a configuration of limit cycles forming two nests.

We also will use a common function in number theory, see [32]. The *partition function* $p : \mathbb{N} \longrightarrow \mathbb{N}$ assigns to each natural number n the number of combinations of positive natural numbers that add up to n. We define $p(0) = 1$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(n)$	1	2	3	5	7	11	15	22	30	42	56	77	101	135

Table 1. Some values of the partition function $p(n)$.

The partition function grows very fast with n; for example $p(50) = 204\,226$, $p(100) = 190\,569\,292$, $p(1000) \approx 2.4 \times 10^{31}$, $p(10000) \approx 3.6 \times 10^{106}$,... In fact, Hardy and Ramanujan, and, independently, Uspensky proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\sqrt{\pi \frac{2n}{3}}\right) \quad \text{as} \quad n \sim \infty.$$

In Niven and Zuckerman's book [32] some recurring expressions for the calculation of $p(n)$ are shown. We highlight, for instance,

$$p(n) = \sum_{j \in \mathbb{Z} \setminus \{0\}} (-1)^{j+1} p\left(n - \frac{1}{2}(3j^2 - j)\right)$$
$$= p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \cdots,$$

where $p(k)$ is taken to be zero when it is evaluated at negative values.

Finally, we also will use *combinations with repetition*. A combination with repetition of k objects taken from a set with n elements is a way of selecting an object in the set, k times in succession, without taking into account the order of the k choices and "with replacement", so the same object can be selected several times. It is well known that there are $\binom{n+k-1}{k}$ combinations with repetition, where $\binom{m}{q}$ is the usual combinatorial number.

3 Fibonacci numbers and limit cycles

In this section we will show how Fibonacci numbers appear when counting the limit cycles stability configurations. Note that since the repelling limit cycles and the semistable ones are both considered unstable, we will not be counting the number of phase portraits when studying the stability configurations.

It is well known (in fact, it is an exercise that appears in some textbooks) that the number of stability configurations of nested limit cycles satisfies the Fibonacci sequence.

Lemma 1. *Let c_n denote the number of stability configurations of a nest of n limit cycles for an analytic planar system* (1). *Then*

$$c_n = c_{n-1} + c_{n-2}, \ \ with \ c_0 = 1 \ and \ c_1 = 2. \tag{2}$$

Hence, $c_n = F_{n+2}$, being F_i the i-th Fibonacci number.

Proof. Suppose that we have n nested limit cycles $\{\gamma_1, \ldots, \gamma_n\}$ forming a nest $\mathcal{N} \subseteq C_{\gamma_n}$ with level n. We consider that $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_n$. These limit cycles can be stable or unstable. We write any stability configuration as $\gamma_1^{k_1} \prec \gamma_2^{k_2} \prec \cdots \prec \gamma_n^{k_n}$ where $k_i \in \{s, u\}$ (below, we also use this notation to indicate the ordering of partial configurations of 2 or 3 cycles).

Observe that if γ_n is unstable then γ_{n-1} can be either stable or unstable, so the following two partial configurations are possible

$$\gamma_{n-1}^u \prec \gamma_n^u \quad \text{and} \quad \gamma_{n-1}^s \prec \gamma_n^u.$$

As a consequence there are as many stability configurations with γ_n^u as stability configurations of $n-1$ limit cycles.

On the contrary, if γ_n is stable then γ_{n-1} can only be unstable, hence we only have the partial stability configuration $\gamma_{n-1}^u \prec \gamma_n^s$. By using the preceding argument we have that the only possible configurations are

$$\gamma_{n-2}^u \prec \gamma_{n-1}^u \prec \gamma_n^s \quad \text{and} \quad \gamma_{n-2}^s \prec \gamma_{n-1}^u \prec \gamma_n^s,$$

so there are as many configurations with γ_n^s as configurations of $n-2$ limit cycles, which proves the relation (2). The initial conditions are also clear: there is only one way to have no limit cycles, so $c_0 = 1$, and, if a unique limit cycle exists, then it can be either stable or unstable and so $c_1 = 2$. $\qquad\square$

We consider now the number of possible stability configurations of two nests of limit cycles each of them surrounding only one of the two singular points P_1 and P_2 with index $+1$ (recall that any limit cycle of a smooth system surrounds at least one singular point). In this case, there can be a nest with i limit cycles surrounding P_1 and another one with j limit cycles surrounding P_2. We denote this class of stability configurations by (i, j). Observe that if there are only two singular points with index $+1$, there cannot exist a nest enveloping both points because, otherwise, there would exist an invariant disk containing only the two fixed points so that the sum of indices would be 2, which would contradict the Poincaré-Hopf theorem, see [13].

Because of the definition of configuration, we identify, and only count as one, the symmetric configurations in the (i, j) and the (j, i) classes, and we will write $(i, j) \sim (j, i)$. More explicitly, consider a case in the class (i, j) such that the stability configuration of the nest surrounding P_1 is $\gamma_1^{k_1} \prec \gamma_2^{k_2} \prec \cdots \prec \gamma_i^{k_i}$ and the stability configuration of the nest surrounding P_2 is $\bar{\gamma}_1^{\ell_1} \prec \bar{\gamma}_2^{\ell_2} \prec \cdots \prec \bar{\gamma}_j^{\ell_j}$, where $k_m, \ell_m \in \{s, u\}$. For reasons of economy, we will write this configuration as

$$(k_1, k_2, \ldots, k_i; \ell_1, \ell_2, \ldots, \ell_j). \tag{3}$$

We will only count as one case the symmetric configurations (3) and

$$(\ell_1, \ell_2, \ldots, \ell_j; k_1, k_2, \ldots, k_i),$$

which belong to the classes (i, j) and (j, i) respectively.

By using (2) we get the following result:

Proposition 1. *The number of stability configurations of n limit cycles surrounding only two singular points of index $+1$ for an analytic planar system* (1) *is*

$$c_{n,2}^* = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} F_{i+2} F_{n-i+2} + \delta(n) \binom{F_{\lfloor n/2 \rfloor + 2} + 1}{2}, \tag{4}$$

where $\delta(n) = (1 + (-1)^n)/2$ and $\lfloor\ \rfloor$ is the floor function.

In Table 2, we present some of the values of $c_{n,2}^*$:

n	1	2	3	4	5	6	7	8	9	10	11	12
$c_{n,2}^{*}$	2	6	11	24	44	86	155	287	510	916	1608	2833

Table 2. Some values of the number of stability configurations of limit cycles surrounding only two points of index $+1$.

Proof of Proposition 1. It is easy to see that, by Lemma 1, the number of configurations in the classes of the form $(i, n-i)$ with $i \neq n-i$, taking into account that $(i, n-i) \sim (n-i, i)$, is

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} c_i c_{n-i} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} F_{i+2} F_{n-i+2}.$$

If n is odd, there are no other cases and Equation (4) holds.

If n is even, we must add the configurations that belong to the class $(n/2, n/2)$. In this case, it is easy to see that if we did not take into account the identification of symmetric cases we would have $c_{n/2}^2$ configurations. But, taking into account the equivalence relation of symmetric cases, and by Lemma 1, we have only

$$\binom{c_{n/2} + 1}{2} = \binom{F_{\lfloor n/2 \rfloor + 2} + 1}{2}$$

configurations, which is precisely the number of combinations with repetition of $c_{n/2}$ elements taken in groups of 2 elements.

Notice that, even though the term $F_{\lfloor n/2 \rfloor + 2}$ in the formula (4) makes sense for n odd, it does not apply in this case because $\delta(n) = 0$. $\square$

Now we consider the case in which, apart from the nests surrounding P_1 and P_2 with levels i and j, there is a third (outer) nest consisting of k cycles surrounding both P_1 and P_2. In this case we say that the system is of class $(i, j; k)$. For instance, the class of the configuration shown in Figure 2 is $(0, 2; 3)$. Again we must identify the symmetric cases in the classes $(i, j; k)$ and $(j, i; k)$. Observe that, from the Poincaré-Hopf Theorem, if $k > 0$ there must be other singular points enveloped by the limit cycles of the outer nest. Moreover, if there are finitely many singular points inside the outer nest, the total sum of their indices must be $+1$ or, equivalently, the sum of the indices out of the nests P_1 and P_2 must be -1.

Proposition 2. *The number of stability configurations of n limit cycles, with configuration $(i, j; k)$ such that $i + j + k = n$, is*

$$c_{n,2} = \sum_{k=0}^{n} \left(\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} F_{i+2} F_{k-i+2} + \delta(k) \binom{F_{\lfloor k/2 \rfloor + 2} + 1}{2} \right) F_{n-k+2}.$$

n	1	2	3	4	5	6	7	8	9	10	11
$c_{n,2}$	4	13	34	82	184	396	821	1659	3277	6362	12163

Table 3. Some values of the number of stability configurations of limit cycles surrounding only two singular points of index $+1$ and some other singular points with total sum of their indices equal to -1.

Some of the values of $c_{n,2}$ are given in Table 3.

Proof of Proposition 2. If we have n limit cycles, we must study the stability configurations of all the classes of the form $(i, k-i; n-k)$ for each $k = 0, \ldots, n$ and $i = 0, \ldots, k$, taking into account that $(i, k-i; n-k) \sim (k-i, i; n-k)$.

By Proposition 1, for each $k \in \{0, \ldots, n\}$, the contribution of the nests that surround P_1 and P_2 to the number of stability configurations is $c_{k,2}^*$. On the other hand, by using Lemma 1, the contribution of the outer nest is $c_{n-k} = F_{n-k+2}$. Therefore the number of configurations for each k is $c_{k,2}^* c_{n-k}$. Adding all the cases we get

$$c_{n,2} = \sum_{k=0}^{n} c_{k,2}^* c_{n-k},$$

so the result follows. □

Obviously, the stability configurations in other nest arrangements can be treated analogously.

4   Number of configurations of ODEs with n limit cycles

In contrast with Section 3, where the stability of the limit cycles plays a key role, in this section we study the maximum number of configurations of limit cycles regarding only its topological distribution. We consider a planar system with exactly n limit cycles. As mentioned in the introduction, the different configurations are in bijection with the different cases of unlabeled rooted trees with $n + 1$ nodes which were studied by Cayley in 1875. Our approach is independent of those we have found in the literature and self-contained. The results that we present are extracted from [21].

We will use some of the notations introduced in Section 2. In particular, recall that if γ is a periodic orbit of our planar system, C_γ denotes the bounded open set with boundary γ. If γ is maximal, C_γ is the domain of γ and the number of limit cycles in $\overline{C_\gamma}$ is the level of γ.

We will first address a simple question such as the number of maximal limit cycles configurations (see thicker cycles in Figures 3 and 4). We can state our first result in terms of the partition function introduced in Section 2.

Lemma 2. *Consider a system of differential equations in the plane with exactly n limit cycles. Then, the maximal limit cycles of this system can be distributed in $p(n)$ different ways.*

Proof. The level of every maximal cycle γ is a positive integer given by the number of limit cycles contained in $\overline{C_\gamma}$. Since the domains of the maximal limit cycles are disjoint, the total number of ways to arrange them in order to have exactly n limit cycles coincides with the number of ways to get the sum of their levels to be n, i.e. $p(n)$. □
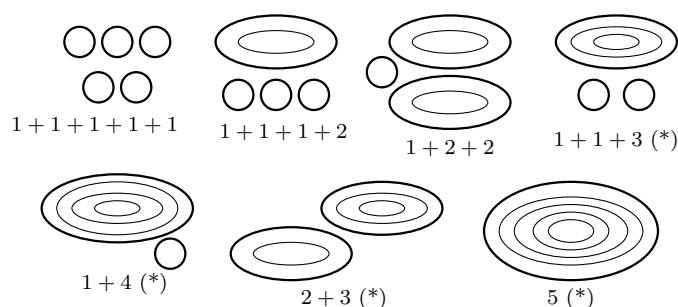

Figure 3. The $p(5) = 7$ different distributions of maximal limit cycles for $n = 5$. Thicker limit cycles are the maximal ones. Maximal limit cycle configurations with (*) give rise to different limit cycle configurations, see Figure 4.

A configuration of maximal limit cycles with α_j domains of level q_j, $j = 1, \ldots, s$ and $q_k \neq q_l$ if $k \neq l$, will be denoted by $(q_1^{\alpha_1}, q_2^{\alpha_2}, \ldots, q_s^{\alpha_s})$. Obviously, for a differential system with the above configuration and exactly n limit cycles it holds that $n = \alpha_1 q_1 + \cdots + \alpha_s q_s$. Notice that, given n there are $p(n)$ different ways of writing n as

$$n = \alpha_{1,k} q_{1,k} + \cdots + \alpha_{s_k,k} q_{s_k,k}, \quad k = 1, 2, \ldots, p(n),$$

with $q_{u,v} \geq 1$ and $\alpha_{u,v} \geq 1$ integer numbers.

Proposition 3. *Consider a system of differential equations in the plane with exactly n limit cycles. Then the maximum number of configurations of limit cycles is given by the recurrent formula*

$$C(n) = \sum_{k=1}^{p(n)} \prod_{i=1}^{s_k} \binom{C(q_{i,k} - 1) + \alpha_{i,k} - 1}{\alpha_{i,k}}, \tag{5}$$

with $C(0) = 1$ for the sake of notation, and $n = \sum_{i=1}^{s_k} \alpha_{i,k} q_{i,k}$ for each $k = 1, 2, \ldots, p(n)$.

The first values of $C(n)$ are given in Table 4. For instance, in Table 4, the value $C(5) = 20$ is obtained by using Proposition 3, by adding the 3 configurations in

Figure 3 without the (*) symbol with the 17 ones given in Figure 4. The whole list of cases is depicted in Figure 1.

n	1	2	3	4	5	6	7	8	9	10
$C(n)$	1	2	4	9	20	48	115	286	719	1842

Table 4. First values of $C(n)$, the number of different configurations of n limit cycles.

Note that, as a consequence of the fact that the sequence $C(n)$ also counts the number of unlabeled rooted trees with $n + 1$ nodes, it is a shifted version of the sequence A000081 in OEIS, [38]. This sequence has been widely studied in connection with other problems (see the comments and references in the above citation). Among the results in the literature, we highlight that the asymptotic expression of this sequence and other properties of its generating function have been studied among others by Pólya in 1937 (see this and other references in [15, p. 72] and [19]).
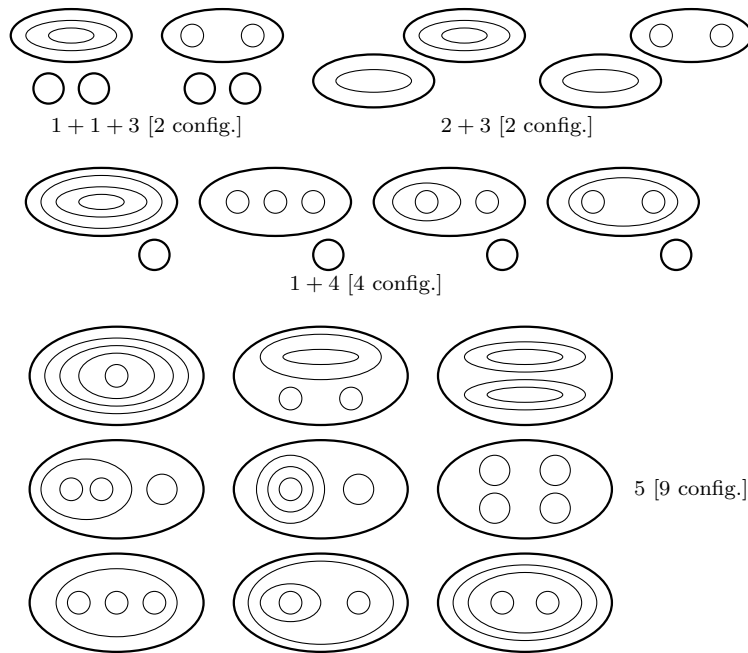

Figure 4. Unfolding of configurations in the case $n = 5$ that have an $(*)$ in Figure 3.

Proof of Proposition 3. We will use the notations and results stated in Lemma 2. Let $(q_1^{\alpha_1}, q_2^{\alpha_2}, \ldots, q_s^{\alpha_s})$ be a partition of n. Since the formula of $C(n)$ is recurrent, it

suffices to see how to calculate $C(n)$ knowing $C(j)$, $j = 1, \ldots, n-1$.

We take a domain with level q. Let γ be its maximal limit cycle. In C_γ there will be exactly $q-1$ limit cycles and hence each domain of level q supports $C(q-1)$ different configurations.

To obtain the total number, we will study how many of them are produced when considering different domains simultaneously. We will distinguish two cases:

(a) Domains have different levels q_A and q_B. In this case, considering them simultaneously we will obtain $C(q_A - 1)C(q_B - 1)$ configurations.

(b) There are α domains of level q. In this case, similarly to Section 3, the total number of configurations cannot be computed simply as $(C(q-1))^\alpha$, since we would count many repetitions. For instance, if we had $\alpha = 2$ nests of level $q = 4$, see Figure 5, then $C(q-1) = C(3) = 4$. Counting all the possible pairs, we would obtain the $(C(3))^2 = 16$ configurations $\{AA, AB, AC, AD, \ldots, DD\}$ while 6 of them are repeated. The correct number of configurations turns out to be $\binom{4+1}{2} = \binom{5}{2} = 10$.

In general, this is a purely combinatorial problem and the number of configurations obtained by joining α domains of level q corresponds to the number of combinations with repetition of $C(q-1)$ elements taken in groups of α elements:
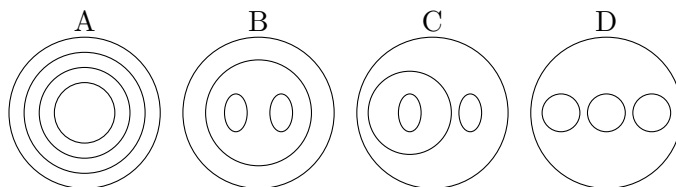$$\binom{C(q-1) + \alpha - 1}{\alpha}.$$

Figure 5. Possible configurations of a domain of level 4 ($q = 4$).

Cases (a) and (b) give us a complete final description: for each partition $(q_1^{\alpha_1}, q_2^{\alpha_2}, \ldots, q_s^{\alpha_s})$, the number of configurations will be given by
$$\prod_{i=1}^{s} \binom{C(q_i - 1) + \alpha_i - 1}{\alpha_i};$$

if we now add all the partitions of n, we arrive to the formula of the statement. □

To end this section we want to make a couple of remarks: (i) any configuration of limit cycles can be realized by a polynomial vector field (with a computable degree), see [29, 31]. In [29] the n limit cycles are hyperbolic while in [31] the limit cycles can also be chosen with any given desired stability and multiplicity; (ii) the extended

Bendixson–Dulac Theorem ([8, 18, 30, 41]) gives, under some conditions, an upper bound of the number of limit cycles of a smooth planar differential system (1) and sometimes even the exact number, allowing to ensure that we are under our main hypothesis: *the system* (1) *has exactly a given number of limit cycles.*

5 Lower bounds for Hilbert numbers

Consider now real planar polynomial systems of ordinary differential equations (1) with P and Q polynomials of degree at most n. We are concerned about the maximum number of limit cycles, $\mathcal{H}(n)$. The knowledge of $\mathcal{H}(n)$ is one of the most elusive problems of the famous Hilbert's list and constitutes the second part of Hilbert's sixteenth problem.

As we have already explained, here we prove the existence of quadratic lower bounds for $\mathcal{H}(n)$ that are obtained following the ideas of [9] and by using very simple background on ODEs and recurrences. The results, as they are presented here, are also developed in [16].

Proposition 4. *There exists a sequence of values n_k tending to infinity and a constant $K > 0$ such that $\mathcal{H}(n_k) > K n_k^2$.*

Proof. The construction of the ODE that gives this lower bound is a recurrent process. Let $X_0 = (P_0, Q_0)$ be a given polynomial vector field of degree n_0 with $c_0 > 0$ limit cycles. Since this number of limit cycles is finite, there exists a compact set containing all of them. Therefore, doing a translation if necessary, we can assume that all of them are in the first quadrant. By simplicity we continue calling X_0 this new translated vector field. From it, we construct a new vector field, by using the (non-invertible) transformation

$$x = u^2, \quad y = v^2.$$

The differential equation associated to X_0 is $\dot{x} = P_0(x,y)$, $\dot{y} = Q_0(x,y)$ and it writes in these new variables as

$$\dot{u} = \frac{P_0(u^2, v^2)}{2u}, \quad \dot{v} = \frac{Q_0(u^2, v^2)}{2v}.$$

By introducing a new time s, defined as $dt/ds = 2uv$, we get that this ODE is transformed into

$$u' = v\, P_0(u^2, v^2), \quad v' = u\, Q_0(u^2, v^2).$$

Since each point lying in the first quadrant (x, y) has four preimages $(\pm\sqrt{x}, \pm\sqrt{y})$, the new ODE has, at each quadrant, a diffeomorphic copy of the positive quadrant of the vector field X_0. Hence this new vector field, which we call X_1, has degree $n_1 = 2n_0 + 1$ and at least $c_1 = 4c_0$ limit cycles. By repeating this process, starting now with X_1 and so on, we get a sequence of vector fields X_k, with respective degrees n_k, having at least c_k limit cycles, where

$$n_{k+1} = 2n_k + 1, \qquad c_{k+1} = 4c_k,$$

see Figure 6.

Solving the above linear difference equation we get that

$$n_k = 2^k(n_0 + 1) - 1, \qquad c_k = 4^k c_0. \tag{6}$$

Hence, since

$$2^k = \frac{n_k + 1}{n_0 + 1} \quad \text{and} \quad 4^k = \frac{c_k}{c_0},$$

we obtain that

$$\frac{c_k}{c_0} = \left(\frac{n_k + 1}{n_0 + 1}\right)^2 > \frac{1}{(n_0 + 1)^2} n_k^2.$$

Consequently,

$$\mathcal{H}(n) > \frac{c_0}{(n_0 + 1)^2} n^2 \quad \text{and} \quad n = 2^k(n_0 + 1) - 1, \quad k \in \mathbb{N},$$

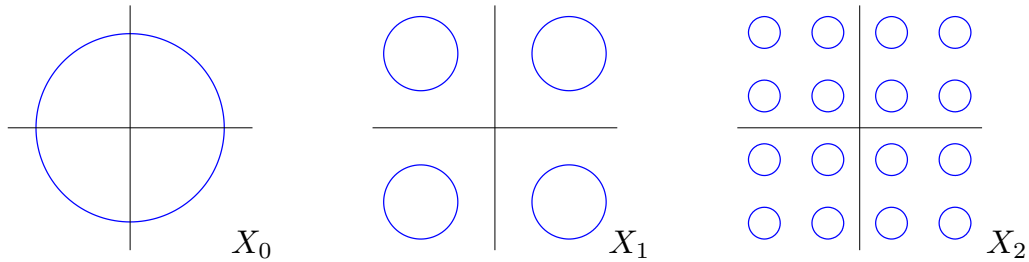as we wanted to prove.   □

Figure 6. Two steps of the construction of the vector fields X_k, starting from a vector field X_0 with a unique limit cycle ($c_0 = 1$, $n_0 = 3$).

Notice that the proof of Proposition 4 shows that the constant K depends on the seed of the procedure. Let us give some examples:

1. Consider X_0 to be the trivial system of ODEs with one limit cycle that, in polar coordinates, writes as

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1.$$

   Clearly, the limit cycle is $r = 1$ and, thus, $c_0 = 1$ and $n_0 = 3$. Hence, $K = 1/16$ and $n_k = 2^{k+2} - 1$.

2. Consider X_0 to be any quadratic system that reveals the lower bound $\mathcal{H}(2) \geq 4$. Then, $c_0 = 4$ and $n_0 = 2$ and, therefore, we get $K = 4/9$ and $n_k = 2^k \cdot 3 - 1$.

n	3	4	5	6	7	8	9	10
h_n	13	28	37	53	74	96	120	142
K	13/16	28/25	37/36	53/49	37/32	32/27	6/5	142/121

Table 5. Lower bounds h_n of $\mathcal{H}(n)$ and the corresponding values of K, for $n \in \{3, \ldots, 10\}$.

3. Consider X_0 to be a system attaining any of the best lower bounds h_n of $\mathcal{H}(n)$ known in the literature, for $n \in \{3, \ldots, 10\}$, see [22–24, 26, 28, 34, 40]. In the third row of Table 5, we get the values of K obtained by this procedure. We remark that the highest K value obtained is 6/5.

We notice that better bounds can be obtained by using finer arguments. Indeed, it is known that $\mathcal{H}(n) \geq L \, n^2 \log(n)$, which is currently the best general result on by lower bounds for $\mathcal{H}(n)$, see again [9] or [1, 26, 27]. In fact, our proof is totally inspired by that of [9], where the authors find additional limit cycles at each step that appear by perturbing the centers created by the method on the axes $uv = 0$. They obtain that, instead of (6), it holds that

$$n_{k+1} = 2n_k + 1, \quad c_{k+1} = 4c_k + (2^k - 2)^2 + (2^k - 1)^2.$$

By studying these new difference equations they get the improved lower bounds of type $L \, n^2 \log(n)$.

6   Maximum number of centers

In this section, following [17], we collect some known results about the maximum number of centers of planar polynomial systems of degree n and we propose an open problem.

The classification of centers for polynomial differential systems started with the quadratic ones with the works of Dulac, Kapteyn at the beginning of the 20th century. It continued with the works of Bautin [4] during the 50's and Sibirskii [36] in the 60's about the cubic systems with homogeneous nonlinearities, including also the study of the cyclicity of the centers.

Proposition 5. *Let $\mathcal{C}_n$ be the maximum number of centers of polynomial differential systems* (1) *of degree n. Then $\mathcal{C}_1 = 1$ and for $n \geq 2$ it holds that*

$$\left\lfloor \frac{n^2 + 1}{2} \right\rfloor \leq \mathcal{C}_n \leq \frac{n^2 + n}{2} - 1.$$

Proof. It is clear that $\mathcal{C}_1 = 1$. Observe that, using Bezout's theorem, a planar polynomial differential system of degree $n > 0$, with finitely many equilibrium points,

has at most n^2 equilibrium points. Moreover, at most $(n^2 + n)/2$ can have index $+1$, see for instance [12, 25].

Therefore, since all centers have index $+1$, it holds that $\mathcal{C}_n \leq (n^2 + n)/2$. The case with infinitely many critical points follows similarly and has a smaller upper bound.

We can refine this upper bound by using the beautiful Euler–Jacobi formula, which asserts, for the planar case, that if a polynomial system $P(x,y) = 0$, $Q(x,y) = 0$, with P and Q having degrees n and m, respectively, has exactly $n\,m$ solutions (hence all them are finite and simple), then

$$\sum_{\{(u,v)\,:\,P(u,v)=Q(u,v)=0\}} \frac{R(u,v)}{\det(\mathrm{D}(P,Q))(u,v)} = 0,$$

for any polynomial $R(x,y)$ of degree smaller than $n + m - 2$, see for instance [20]. Here, $\mathrm{D}(P,Q)$ denotes the differential of the map (P,Q). As consequence of the Euler–Jacobi formula, one obtains that, for $n \geq 2$, if all points with index $+1$ lie on a single algebraic curve, then the curve must have degree strictly greater than $n-1$, see [10]. Having into account that all centers lie on the algebraic curve $\mathrm{div}(P,Q) = 0$, which has at most degree $n - 1$, we get that $\mathcal{C}_n \leq (n^2 + n)/2 - 1$.

On the other hand, in [11] it is also proved that planar polynomial *Hamiltonian* differential systems of degree n have at most $\lfloor (n^2 + 1)/2 \rfloor$ centers and, hence, $\mathcal{C}_n \geq \lfloor (n^2 + 1)/2 \rfloor$. Moreover, this upper bound is attained: consider, for instance,

$$\dot{x} = F(y), \quad \dot{y} = -F(x), \quad \text{with} \quad F(u) = \prod_{j=1}^{n}(u - j);$$

for these systems, centers and saddles are placed like white and black squares on a $n \times n$ chessboard. This concludes the proof. $\qquad\square$

From Proposition 5, then, we know that $\mathcal{C}_2 = 2$, $\mathcal{C}_3 = 5$, and $8 \leq \mathcal{C}_4 \leq 9$. Therefore, it is natural to consider the following open problem:

> *Determine the maximum number of centers, $\mathcal{C}_n$, for planar polynomial differential systems of degree $n \geq 4$.*

Once the number $\mathcal{C}_n$ is obtained, it is also interesting to know the different possible phase portraits that systems having these maximal number of centers can have, see for instance [6], where the Hamiltonian case when $n = 3$ is studied. One of the reasons is that perturbations of these systems are good candidates to have different configurations of limit cycles.

We end this section with some comments for another family, the planar polynomial holomorphic systems of degree n, for which the full classification of the different phase portraits with the maximum number of centers is known. Recall that these systems are the ones that in complex coordinates $z = x + \mathrm{i}y$ write as $\dot{z} = p_n(z)$, being p_n a complex polynomial of degree n. For them, the maximum number of

centers is n and, in this case, also the maximum number $N(n)$ of different topological phase portraits on the Poincaré disc can be computed, see [2, 35]. It turns out that $N(n)$ coincides with the number of labeled projective planar trees with n nodes and is given by the sequence A006082 in the OEIS' web page [39]. In particular, $N(4) = 2$, $N(5) = 3$ and $N(6) = 6$, see the 6 different classes of phase portraits for $n = 6$ in Figure 7. In these phase portraits each point is a center and the connected component where this point lies is filled of periodic orbits surrounding it.
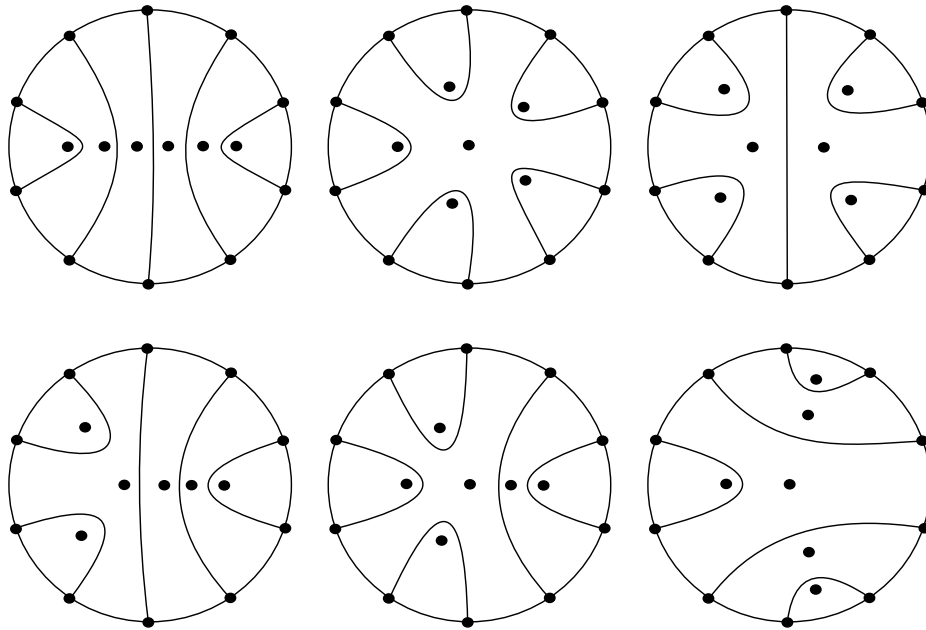

Figure 7. The six different phase portraits with six centers for holomorphic vector fields of degree 6.

Acknowledgements

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