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Criteria for the nonexistence of periodic orbits in planar differential systems

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Abstract. In this work we summarize some well-known criteria for the nonexistence of periodic orbits in planar differential systems. Additionally we present two new criteria and illustrate with examples these criteria.

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1 Introduction and statement of the main results

We consider a *planar differential system* that we write as

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are C^1 real functions in the variables x and y, and t is the independent variable.

The objective of this note is double, first we recall the more well-known results for the nonexistence of periodic orbits of a differential system (1). Second we provide two new criteria for the nonexistence of periodic orbits of system (1).

As far as we know one of the first criterium of nonexistence is the following one due to Poincaré.

Theorem 1 (Poincaré Method of Tangential Curves)**.** *Consider a family of curves $F(x, y) = C$, where $F(x, y)$ is continuously differentiable. If in a region R the quantity*

$$\frac{dF}{dt} = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y}$$

has constant sign, and the curve

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} = 0$$

(which represents the locus of points of contact between curves in the family and the trajectories of (1)*, and is called a tangential curve) does not contain a whole trajectory of* (1) *or any closed branch, then system* (1) *does not possess a periodic orbit which is entirely contained in R.*

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For a proof of Theorem 1 see either Theorem 1.9 of [8], or Proposition 7.9 of [3].

Theorem 2 (Bendixson's Theorem). *Assume that the divergence function $\partial P/\partial x + \partial Q/\partial y$ of system* (1) *has constant sign in a simply connected region R, and is not identically zero on any subregion of R. Then system* (1) *does not have a periodic orbit which lies entirely in R.*

For a proof of Theorem 2 see either Theorem 1.10 of [8], or Section 3.9 of [6], or Proposition 1.133 of [2], or Theorem 7.10 of [3].

Theorem 3 (Dulac's Theorem). *If for system* (1) *there exists a C^1 function $B(x,y)$ in a simply connected region R such that $\partial(BP)/\partial x + \partial(BQ)/\partial y$ has constant sign and is not identically zero in any subregion, then this system* (1) *does not have a periodic orbit lying entirely in R.*

For a proof of Theorem 3 see either Theorem 1.12 of [8], or Theorem 4.8 of [9], or Section 3.9 of [6], or Exercise 1.136 of [2], or Theorem 7.12 of [3].

The well-know Liénard differential equation [4]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x)$ and $g(x)$ are C^1 functions in the open subset R of $\mathbb{R}^2$, can be written as the following first order differential system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)y. \tag{2}$$

Theorem 4 (Chen–Yang–Zhang–Zhang's Theorem). *Assume that the differential system* (2) *satisfies the following conditions:*

(i) $g(x) = -g(-x)$ *and* $xg(x) > 0$ *if* $x \neq 0$*;*

(ii) $f(x) = f_1(x) + f_2(x)$ *with* $f_1(x) = f_1(-x)$, $f_2(x) = -f_2(-x)$ *and* $f_1(x) \neq 0$.

Then this system (2) *has no periodic orbits in R.*

Theorem 4 is a particular case of Theorem 1 of [1].

As far as we know the next two criteria for the nonexistence of periodic orbits are new.

Let $f(x,y) = 0$ be a curve, then a point (x_0, y_0) of this curve is a contact point with system (1) if it satisfies $(P\partial f/\partial x + Q\partial f/\partial y)(x_0, y_0) = 0$.

Theorem 5 (Transversal divergence criterium). *Let $D(x,y) = \partial P/\partial x + \partial Q/\partial y$ be the divergence of system* (1)*. If the curve $D(x,y) = 0$ has no contact points of even multiplicity with the system* (1)*, then this system has no periodic orbits.*

The proof of Theorem 5 is given in Section 2.

Working in polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$ system (1) writes

$$\dot{r} = \left.\frac{xP + yQ}{\sqrt{x^2 + y^2}}\right|_{(x,y)=(r\cos\theta, r\sin\theta)}, \quad \dot{\theta} = \left.\frac{xQ - yP}{x^2 + y^2}\right|_{(x,y)=(r\cos\theta, r\sin\theta)}.$$

Theorem 6 (Angular velocity criterium)**.** *Assume that the origin of coordinates is an equilibrium point of a system* (1)*, and that the component γ of the curve $xQ - yP = 0$ passes through the origin of coordinates and locally on one side of this curve we have $xQ - yP > 0$ and on the other side $xQ - yP < 0$. Then system* (1) *has no periodic orbits surrounding the origin crossing the component γ at a point with odd mutiplicity.*

Theorem 6 is proved in Section 3.

2 Proof of Theorem 5

By the Bendixson Theorem any periodic orbit of system (1) must intersect the curve $D(x, y) = 0$. But under the assumptions of Theorem 5 the flow of this system is transversal at all the point of the curve except at its possible contact points of odd multiplicity, but also at these points the flow crosses the curve $D(x, y) = 0$. Hence clearly a periodic orbit cannot intersect the divergence curve $D(x, y) = 0$ and consequently it does not exists. This completes the proof of Theorem 5.

Now we present an application of Theorem 5. We consider the Selkov-Higgins system which is relevant in the study of the glycolysis. This system when one of its parameters is equal to 2 writes

$$\dot{x} = 1 - xy^2 = P(x, y), \qquad \dot{y} = ay(xy - 1) = Q(x, y). \tag{3}$$

The divergence of this system is $D(x, y) = -a + 2axy - y^2$. Now we study the transversality of the flow of system (1) on the curve $D(x, y) = 0$, that is

$$p(y) := \frac{\partial D}{\partial x} P + \frac{\partial D}{\partial x} Q \Big|_{D=0} = \frac{1}{2}(-a^2 + 4ay - 3y^4).$$

Using the formulas of Lu Yang [7] for this quartic polynomial we have

$$D_2 = 0, \quad D_3 = -2592a^2, \quad D_4 = 6912a^4(a^2 - 9).$$

When $|a| > 3$ then $D_4 > 0$ and $D_3 \leq 0$ or $D_2 \leq 0$, and the polynomial $p(y)$ has no real roots. Consequently by Theorem 5 system (3) has no periodic orbits.

If $a = \pm 3$ then $D_4 = 0$ and $D_3 < 0$, and the polynomial $p(y)$ has one double real root. So system (3) again by Theorem 5 has no periodic orbits.

If $a = 0$ then $D_4 = D_3 = D_2 = 0$ and the polynomial $p(y)$ has one quadruple real root. Hence by Theorem 5 system (3) has no periodic orbits.

Finally if $a \in (-3, 0) \cup (0, 3)$ then $D_4 < 0$, and the polynomial $p(y)$ has two real simple roots. In this case we cannot apply Theorem 5 and system (3) could have periodic orbits for some of these values of a. Indeed in the work [5] values of $a \in (-3, 0) \cup (0, 3)$ are given for which system (3) has periodic orbits.

3 Proof of Theorem 6

Assume that there exists a periodic orbit Γ surrounding the origin which crosses the component γ at a point p with odd multiplicity. Then on this periodic orbit Γ in a neighborhood of p and on one side of γ we have $\dot{\theta} > 0$ and on the other side $\dot{\theta} < 0$, this provides a contradiction because in a small neighborhood of p the periodic orbit must be either $\dot{\theta} \geq 0$, or $\dot{\theta} \leq 0$. This completes the proof of Theorem 6.

Now we present one application of Theorem 6. Consider the differential system

$$\dot{x} = -x(2 + f(x, y)) + y = P(x, y), \qquad \dot{y} = -y(2 + f(x, y)) + x = Q(x, y), \tag{4}$$

where the C^1 function f is such that it, its first and second derivatives vanish at the origin of coordinates. Then the origin of coordinates is a stable node with eigenvalues -1 and -3, and $xQ - yP = x^2 - y^2$, and consequently $\dot{\theta} = 0$ is formed by the two straight lines $y = \pm x$. So by Theorem 6 system (4) cannot have periodic orbits surrounding the origin.

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