

A survey on local integrability and its regularity

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Abstract. In this survey paper, we summarize our results and also some related ones on local integrability of analytic autonomous differential systems near an equilibrium. The results are on necessary conditions related to existence of local analytic or meromorphic first integrals, on existence of analytic normalization of local analytically integrable system, and also on some sufficient conditions for existence of local analytic first integrals. Among which the results are also on regularity of the local first integrals, including analytic and Gevrey smoothness. We also present some open questions for further investigation.

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1. Introduction

For analytic autonomous differential system

$$\dot{y} = \frac{dy}{dx} = F(y), \quad y \in \Omega \subset \mathbb{R}^n, \quad (1)$$

where Ω is an open domain, the problem on existence of local or global first integrals is classical. This problem can be traced back to Poincaré [17] and Darboux [3, 4]. A smooth function $H(y)$ is a first integral of system (1) on Ω if $\langle \nabla H, F(y) \rangle \equiv 0$ except perhaps a zero Lebesgue measure subset. Hereafter $\langle \cdot, \cdot \rangle$ represents the inner product of two vectors in \mathbb{R}^n . System (1) is analytic (smooth) integrable if it has $n - 1$ functionally independent analytic (smooth) first integrals. Here our system is autonomous, we consider the first integrals only depending on the dependent variables, because our aim is to apply these first integrals to describe the dynamics of the system in the phase space. Of course, we can consider first integrals including also the independent variable. Since we want to study orbits in the phase space, we consider here only the first integrals in the phase variables y .

For a given analytic system (1), as it is well known that if $y = y_0$ is a regular point, system (1) is analytically integrable around y_0 , i.e. it has $n - 1$ functionally independent local analytic first integrals around y_0 . This argument can be verified using the flow-box theorem or proved directly using the solution of the initial value problem with the fixed initial time x_0 and the initial values near y_0 . When $y = y_0$ is a singular point of system (1), the problem on existence of functionally independent

analytic or smooth first integrals becomes very difficult. In this case we can write system (1), after possibly a translation, in the form

$$\dot{y} = Ay + f(y), \quad f(y) = O(|y|^2), \quad (2)$$

where without loss of generality we can assume that A is in Jordan normal form. Let \mathcal{Y} be the vector field associated to this differential system, and set $\mathcal{Y} = \sum_{j=1}^{\infty} \mathcal{Y}_j$, where \mathcal{Y}_j is the j th homogeneous part of \mathcal{Y} .

As we will show that the existence and number of functionally independent first integrals of system (2) in a neighborhood of the origin are strongly related to the eigenvalues and their relations. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the n -tuple of eigenvalues of A . We say that λ is resonant if

$$\mathcal{R} := \{m \in \mathbb{Z}_+^n \mid \langle m, \lambda \rangle = 0, |m| \geq 2\} \neq \emptyset,$$

where \mathbb{Z}_+ is the set of nonnegative integers, and $|m| = m_1 + \dots + m_n$ for $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$. The classical Poincaré theorem on integrability [17] states that the analytic differential system (2) has no analytic or formal first integrals in a neighborhood of the origin provided that the eigenvalues of A are not resonant. See also [8, 21].

According to the classical result mentioned above by Poincaré, in order that system (2) has an analytic first integral or a formal first integral near the origin, the eigenvalues of A must be resonant.

1 Necessary conditions on existences of first integrals

In 2008, Chen, Yi and Zhang [2] obtained some necessary conditions on existence and number of functionally independent analytic or formal first integrals.

Theorem 1. *Assume that the number of \mathbb{Q}_+ -linearly independent elements of \mathcal{R} is m . Then the analytic differential system (2) has at most m functionally independent analytic or formal first integrals in a neighborhood of the origin.*

The proof depends on the inductive calculations on analytic or formal first integral of the form

$$H(y) = \sum_{j=q}^{\infty} H_j(y)$$

with H_j homogeneous polynomials of degree j , via

$$\begin{aligned} \mathcal{L}(H_q)(y) &:= \langle \nabla H_q, Ay \rangle = 0, \\ \mathcal{L}(H_j)(y) &= - \sum_{s=2}^{j-q+1} \mathcal{Y}_s(H_{j-s+1}), \quad j = q+1, \dots \end{aligned}$$

by inductive calculations together with the invertibility of the linear operator \mathcal{L} on each linear space formed by homogeneous polynomials of any given degree. For all functionally independent analytic or formal first integrals H_1, \dots, H_m of system (2), one can assume without loss of generality [34, 35] that their lowest parts $H_1^0(y), \dots, H_m^0(y)$ are functionally independent. For a proof, see e.g.[9]. Then the problem is turned to the maximum number of functionally independent monomial solutions of $\mathcal{L}(H_q)(y) = 0$. The solutions of this problem is equivalent to the spectrum of the linear operator

$$\mathcal{L}(H_\ell)(y) = \nabla H_\ell(y) Ay$$

on the linear space $\mathcal{H}_\ell(y)$, formed by the homogeneous polynomial of degree ℓ in the n variables y . By [12,31] the spectrum of \mathcal{L} on $\mathcal{H}_\ell(y)$ is $\mathcal{R}_\ell := \{m \in \mathbb{Z}_+^n \mid \langle m, \lambda \rangle, |m| = \ell\}$.

In 2007 Shi [20] extended the Poincaré’s result to the existence of meromorphic first integrals of the analytic differential system (2). Cong, Llibre and Zhang [6] further developed Shi’s result to the version of Theorem 1 on the number of functionally independent meromorphic first integrals.

Theorem 2. *Assume that*

$$\mathcal{R}_Q := \{m \in \mathbb{Z}^n \mid \langle m, \lambda \rangle = 0, |m| = |m_1| + \dots + |m_n| \geq 2\}$$

contains r number of \mathbb{Q} -linearly independent elements. Then system (2) has at most r functionally independent meromorphic first integrals.

The proof adopts the ideas from those of Theorem 1. For functionally independent meromorphic first integrals

$$H_1(y) = \frac{P_1(y)}{Q_1(y)}, \quad \dots, \quad H_r(y) = \frac{P_r(y)}{Q_r(y)}$$

one can assume without loss of generality that the lowest order terms

$$H_1^0(y) = \frac{P_1^0(y)}{Q_1^0(y)}, \quad \dots, \quad H_r^0(y) = \frac{P_r^0(y)}{Q_r^0(y)}$$

are functionally independent. For a proof see e.g.[6, Lemma 6], otherwise one can take polynomials $W_j(z_1, \dots, z_j)$ such that

$$H_1(y), W_2(H_1, H_2), \quad \dots, \quad W_r(H_1, \dots, H_r)$$

have their lowest order rational homogeneous parts being functionally independent. Here the lowest order rational homogeneous part of a meromorphic function $H(y) = P(y)/Q(y)$ with P, Q analytic and having the expansions $P(y) = P^0(y) + \text{h.o.t}$ and $Q(y) = Q^0(y) + \text{h.o.t}$ is $P^0(y)/Q^0(y)$, because after expansion

$$H(y) = \frac{P(y)}{Q(y)} = \frac{P^0(y)}{Q^0(y)} + \sum_{j=1}^{\infty} \frac{A^j(y)}{B^j(y)}$$

one has

$$\deg P^0(y) - \deg Q^0(y) < \deg A^j(y) - \deg B^j(y), \quad \text{for all } j \geq 1$$

where A^j, B^j 's are homogeneous polynomials of degree j . Each $\frac{A^j(y)}{B^j(y)}$ is a rational homogeneous function. The next proofs can be down in a similar way as those in the proof of Theorem 1, by considering the linear operator defined by the linear part of system (2) acting on the set of the lowest order rational homogeneous parts of the meromorphic first integrals.

Associated to Theorems 1 and 2, Llibre, Walcher and Zhang [16] provided a version on local Darboux first integrals of analytic differential systems via Poincaré-Dulac normal form.

Theorems 1 and 2 establish only the necessary conditions on the existence of analytic first integrals. As we know, it is really difficult to provide a sufficient condition on existence of analytic first integrals. The typical one is the center-focus problem in the general case. This is to characterize planar analytic differential systems which have an equilibrium with a pair of pure imaginary eigenvalues. Of course, in this case the two eigenvalues are \mathbb{Z}_+ -resonant. But the problem whether it admits an analytic first integral was solved only for quadratic differential systems. See example [18, 19, 31].

2 Analytic normalization of local analytically integrable differential systems

As it is well known, the sufficient condition for existence of functionally independent first integrals is hard to be found for general planar analytic even polynomial differential systems. Sometimes the equivalent characterization to analytic integrability of analytic differential systems is helpful to determine local properties of the system near an equilibrium. See for instance the next result by Poincaré.

To state the next result, we recall the definitions on Poincaré-Dulac normal form and resonant terms. For system (2) in \mathbb{R}^n with A in Jordan normal form, if the n -tuple of eigenvalues λ of A have complex conjugate ones, saying for example λ_j and $\lambda_{j+1} = \overline{\lambda_j}$, whose associated variables are y_j and y_{j+1} , we set $z_j = y_j + \sqrt{-1}y_{j+1}$ and $z_{j+1} = y_j - \sqrt{-1}y_{j+1}$. For each real eigenvalue λ_s , whose associated variable y_s is replaced by z_s . In these new coordinates, system (2) becomes

$$\dot{z} = Bz + g(z),$$

with B in lower triangular matrix. Its Poincaré-Dulac normal form system is the one

$$\dot{z} = Bz + h(z),$$

with $h(z)$ consisting of the resonant monomials. A monomial $z^k e_l$ in $h(z)$, $k \in \mathbb{Z}_+^n$, $l \in \{1, \dots, n\}$, is resonant, if $\langle k, \lambda \rangle = \lambda_l$, where e_l is the unique vector whose l th component is 1 and all others are 0.

In the next results, the Poincaré-Dulac normal form system is of the special form

$$\dot{z} = Bz(1 + \rho(z)), \quad (3)$$

where $\rho(z)$ consists of the monomials of the form z^k satisfying $\langle k, \lambda \rangle = 0$. We also call the monomials in $\rho(z)$ resonant ones. In the two dimensional case with complex eigenvalues, if the eigenvalues are resonant, they must be conjugate pure imaginary ones. Then each resonant monomial in $\rho(z)$ must be of the form $z^k = z_1^{k_1} z_2^{k_2} = (y_1^2 + y_2^2)^{k_1}$, which is real. So when we write system (3) again in real coordinates $y = (y_1, y_2)$, one gets

$$\dot{y} = Ay(1 + \rho(y))$$

with $\rho(y)$ consisting of resonant monomials, which are powers of $y_1^2 + y_2^2$.

Now we can state the next result.

Theorem 3. *Assume that system (2) is two dimensional and A has a pair of pure imaginary eigenvalues. Then the following statements hold.*

- (a) *The origin is a center if and only if the system is analytically equivalent to its Poincaré-Dulac normal form*

$$\dot{y} = Ay(1 + \rho(y)),$$

where $\rho(y)$ is an analytic function in $y_1^2 + y_2^2$. That is, $\rho(y)$ consists of resonant monomials.

- (b) *The origin is a center if and only if the system has an analytic first integral of the form $y_1^2 + y_2^2 + h.o.t.$*

Statement (b) is useful in studying the center-focus problem. For a two dimensional polynomial differential system of form (2) with the origin having a pair of pure imaginary eigenvalues, one tries to find its Lyapunov quantities of the system at the origin. According to the Hilbert's basis theorem, among the Lyapunov quantities there are only finitely many ones being independent, all the others are functions of these finite ones. If these finite number of Lyapunov quantities vanish, then all Lyapunov quantities vanish, and so the origin is a center. But the Hilbert's basis theorem does not provide a technique to compute this number. And using mathematical softs one can compute only a small number of Lyapunov quantities. Setting these Lyapunov quantities to be zero provides some conditions on the coefficients of the system. Under these coefficient conditions, if we can find an analytic first integral defined in a neighborhood of the origin, then the origin must be a center.

This classical result by Poincaré was extended to higher dimensional analytic differential systems. We summarize the results on this kind of generalization. For analytically integrable Hamiltonian systems in the Liouvillian sense, Ito [9,10] proved the convergence of a symplectic normalization which sends the Hamiltonian system to its Birkhoff normal form under a so-called strong one resonant condition on the eigenvalues of the linearized system at the equilibrium. Zung [36] in 2005 proved

in general that any analytically integrable differential system in Liouvillian sense is analytically equivalent to its Birkhoff normal form via torus action. Ito [11] extended further these results to superintegrable Hamiltonian systems. On the normal form theory, we refer the readers to Bibikov [1], Chow et al [5], Li [12] and Zung [36, 37] for more information on the general definitions and results.

Beside integrable Hamiltonian systems, Zung [37] in 2002 characterized the convergence of the normalization of analytically integrable differential system to its Poincaré-Dulac normal form via torus action. He did not present the concrete representations of the normal form systems. In 2008, Zhang [28] proved the existence of analytic normalization of an analytically integrable differential system to its Poincaré-Dulac normal form under the assumptions that the origin is non-degenerate, and that the matrix A is diagonal. In 2013 Zhang [29] further released these restrictions and obtained the next results.

Theorem 4. *Assume that the n -tuple of eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$ is not zero, i.e. A has at least one eigenvalue not equal to zero. Then system (2) is analytically integrable at the origin, i.e. it has $n - 1$ functionally independent analytic first integrals in a neighborhood of the origin if and only if the resonant set \mathcal{R} has $n - 1$ \mathbb{Q}_+ linearly independent elements and system (2) is analytically equivalent to its distinguished normal form*

$$\dot{y} = \text{diag}(\lambda_1, \dots, \lambda_n)y(1 + g(y))$$

by a near identity analytic normalization, where $g(y)$ has no a constant term and is an analytic function in all resonant monomials y^m with $m \in \mathcal{R}$ being \mathbb{Q}_+ linearly independent elements.

Hereafter \mathbb{Q}_+ and \mathbb{Z}_+ are respectively the sets of nonnegative rational numbers and of nonnegative integers.

The proof of sufficiency is very easy, but the proof of necessity is relatively complicated. The proofs are separated in several steps. The first step is to get the Poincaré-Dulac normal form, the second step is to prove that the analytic or formal first integrals of the Poincaré-Dulac normal form system consist of resonant monomials. By analytic integrability of the original system, one gets in the third step the special form of the normal form system as mentioned in the theorem, and that the eigenvalues of A does not satisfy the small divisor condition. The fourth step is to present the concrete expressions of the coefficients of the normalization and $g(y)$ in terms of the coefficients of $f(y)$ in (2), and use them to prove the convergence of the normalization and $g(y)$ by the majorant series and the implicit function theorem.

We remark that Zhang [29] in 2013 also established a version of Theorem 4 for analytically integrable diffeomorphisms, where the normal form system has a much involved structure. Some related results to Theorem 2 can also be founded in [15] by Llibre, Pantazi and Walcher. In [16], Llibre et al studied also the effect of local Darboux integrability on existence of analytic normalizations.

On the relation between analytic normalization and analytic integrability of analytic differential systems near an equilibrium, Wu and Zhang [24] extended these results near an equilibrium to a periodic orbit of analytic autonomous differential systems. Du, Romanovski and Zhang [7] further developed the above results to partly integrable analytic differential systems.

The known results on characterization of local integrability near an equilibrium were obtained by using the resonance of the eigenvalues of the linearized matrix of the analytic differential system at the equilibrium. It is possible to obtain some necessary conditions on analytic or meromorphic integrability by using the higher order terms of the systems.

Open problem 1. How to apply the higher order terms of analytic differential systems, beside their linear parts, to obtain more necessary conditions on existence of sufficient number of functionally independent analytic or meromorphic first integrals?

Because any smooth function of first integrals is also a first integral, this causes difficulty in finding first integrals of a given analytic differential systems. Du, Romanovski and Zhang [7] in 2016 provided the next result on the structure of first integrals, which is very useful in finding the first integrals of a given analytic differential system.

Theorem 5. *Let \mathcal{Y} be the analytic vector field associated to system (2).*

(a) *There exists a series $\Psi(y)$ such that*

$$\mathcal{Y}(\Psi)(y) = \sum_{m \in \mathcal{R}} v_m y^m$$

where the sum takes over all possible resonant elements in \mathcal{R} . v_m 's are polynomials in the coefficients of \mathcal{Y} , and are called integrable varieties.

(b) *If the vector field \mathcal{Y} has ℓ functionally independent analytic or formal first integrals, then \mathcal{Y} has ℓ functionally independent first integrals of the form*

$$H_1(y) = \alpha_1 y^{m_1} + h_1(y), \dots, H_\ell(y) = \alpha_\ell y^{m_\ell} + h_\ell(y),$$

where $\alpha_1, \dots, \alpha_\ell \in \mathbb{R}$, $m_1, \dots, m_\ell \in \mathbb{Z}_+^n$ are \mathbb{Q}_+ -linearly independent elements of \mathcal{R} , and each h_j is composed of nonresonant monomials in y of degree larger than m_j .

We remark that in statement (a), the series $\Psi(y)$ could consist of both resonant monomials and nonresonant monomials, but its resonant monomials could be arbitrarily chosen. Here a resonant monomial y^m is the one with $m \in \mathcal{R}$.

According to this result, we want to know whether there is a corresponding version to Darboux polynomials of a polynomial differential system. If yes, it will bring great simplification on the searching of Darboux polynomials, because any product of Darboux polynomials is also a Darboux polynomial. Recall that for a

polynomial vector field $\mathcal{P}(y)$, a polynomial $f(y)$ is a Darboux polynomial of \mathcal{P} if there exists a polynomial $k(y)$ such that

$$\mathcal{P}(f)(y) = k(y)f(y),$$

with $k(y)$ a cofactor of $f(y)$.

Open problem 2. For polynomial differential systems, is there a result similar to Theorem 5(b) on Darboux polynomials?

Recall that Darboux polynomials played an important role in characterizing local and global dynamics of polynomial differential systems. For instance, if a polynomial differential system has a sufficient number of Darboux polynomials, say f_1, \dots, f_p , with the corresponding cofactors k_1, \dots, k_p satisfying

$$c_1 k_1(y) + \dots + c_p k_p(y) = 0$$

with $c_1, \dots, c_p \in \mathbb{C}$, then the polynomial differential system has a Darboux first integral

$$H(y) = f_1^{c_1}(y) \cdot \dots \cdot f_p^{c_p}(y).$$

As we know [31], if a quadratic differential system has an equilibrium as a center, then the system has a Darboux first integral defined in a neighborhood of the equilibrium.

The first two sections provide some known results on necessary conditions for existence of functionally independent analytic or formal or meromorphic first integrals of analytic differential systems, or on the equivalent characterization for existence of analytic normalization via analytic integrability of analytic differential systems. Next we recall some results on sufficient conditions for existence of a given type of first integrals of analytic differential systems near an equilibrium.

3 Sufficient condition on existence of local first integrals

In this direction, there are lots of results, especially on center-focus problem for concrete planar polynomial differential systems. But there are seldom systematic results on general analytic differential systems. As we mentioned previously, in order that system (2) have an analytic or a formal first integral around the origin, the eigenvalues of A must be resonant. The simplest resonances are the cases that there are two eigenvalues, say λ_1, λ_2 , satisfying $\lambda_1/\lambda_2 = -1$ and the others are nonresonant, and that one of the eigenvalues is zero, and the others are nonresonant. The former is on the center-focus problem. As we know, there are lots of published papers related to the center-focus problem. And also there have appeared many published books summarizing these results. See e.g. Liu, Li and Huang [14], Romanovski and Shafer [18], Ye [26, 27] and Zhang et al [33]. But as we knew, the center-focus problem is far from being solved, even for cubic differential systems.

Here we recall the results on the latter, which has one zero eigenvalue and the others are nonresonant. This work was initiated from 2003 by Li, Llibre and Zhang

[13]. As before, let $\lambda = (\lambda_1, \mu)$, with $\mu = (\lambda_2, \dots, \lambda_n)$, be the n -tuple of eigenvalues of A . Set

$$\mathcal{R}_0 := \{m \in \mathbb{Z}_+^{n-1} \mid \lambda_1 = 0, \langle \mu, m \rangle = 0, |m| \geq 2\}.$$

The main results in [13] are the following.

Theorem 6. *Assume that a system of form (2) is analytic and that \mathcal{R}_0 is empty, i.e. μ does not satisfy any \mathbb{Q}_+ -resonant condition. The following statements hold.*

- (a) *When $n = 1, 2$, system (2) has an analytic first integral in a neighborhood of the origin if and only if the equilibrium $y = 0$ is not isolated. That is, system (2) has a curve passing the origin, which is full of equilibria.*
- (b) *When $n > 2$, system (2) has a formal first integral around the origin if and only if the equilibrium $y = 0$ is not isolated.*

In one-dimensional case, there exists a unique eigenvalue, which is zero. So the existence of analytic first integral implies that the system is trivial.

In two-dimensional case, the existence of analytic first integral forces by Theorem 5(b) the existence of analytic first integral of the form

$$H(y) = y_1 + h_1(y)$$

with h_1 consisting of higher order terms. Then after the near identity transformation $z = (z_1, z_2) = \beta(y) := (y_1 + h_1(y), y_2)$, system (2) is equivalently changed to

$$\dot{z}_1 = 0, \quad \dot{z}_2 = \dot{y}_2 = \lambda_2 z_2 + f_2(\beta^{-1}z)$$

Clearly, this last system has the analytic curve $\lambda_2 z_2 + f_2(\beta^{-1}z) = 0$ being full of equilibria, and consequently system (2) has a curve filled up with equilibria. Conversely, $\lambda_2 y_2 + f_2(y) = 0$ has an analytic solution, saying $y_2 = \zeta_2(y_1)$, such that $f_1(y_1, \zeta_2(y_1)) \equiv 0$ in a neighborhood of the origin. Then the original system can be written in

$$\dot{y}_1 = (y_2 - \zeta_2(y_1))g_1(y), \quad \dot{y}_2 = \lambda_2(y_2 - \zeta_2(y_1))g_2(y).$$

Comparing with the original system gives that $g_2(y) = 1 + \text{h.o.t.}$ and this last system has the same first integral as that of

$$\dot{y}_1 = g_1(y), \quad \dot{y}_2 = \lambda_2 g_2(y).$$

Its origin is regular, and so it has an analytic first integral near the origin. Hence the original system has an analytic first integral in a neighborhood of the origin.

For higher dimensional system, the proof of (b) follows from the Poincaré-Dulac normal form via a sufficiently higher order cut of the formal transformation, and the order of an isolated equilibrium is independent of the choice of a near identity analytic transformation.

After this result was published in 2003, a long time has passed in before we could determine whether the formal first integral in statement (b) of Theorem 6(b) could be replaced by some first integrals with suitable regularity. Zhang [30] in 2017 answered this problem under suitable conditions on the nonresonant eigenvalues of A .

Theorem 7. *Assume that the elements of μ either all have positive real parts or all have negative real parts. Then system (2) has an analytic first integral at the origin if and only if the equilibrium $y = 0$ is not isolated.*

The proof of the necessity follows from statement (b) of Theorem 6, or can be proved independently using the result in Theorem 5(b). The sufficiency could be proved using the existence of an analytic invariant manifold along the curve filled up with the equilibria and the normal form system.

Theorem 7 has a C^∞ version, see Theorem 2 of Zhang [30]. But when the eigenvalues μ have both positive real parts and negative real part, as shown in Theorem 3(b₂) of [30], there exist systems of form (2) which have no analytic first integrals defined in a neighborhood of the origin. Then one has to ask: under this last condition, does system (2) have C^∞ first integrals in a neighborhood of the origin. Zhang [32] had worked on this problem.

Recently, this kind of study has been developed to Gevrey systems of form (2) with μ nonresonance, see [25]. For more information on Gevrey smoothness, see e.g. Stolovitch [22] and Wu et al [23]. As usual, the classes C^∞ or \mathcal{G}_s for $s \geq 1$ denote the sets of functions C^∞ or Gevrey- s smooth. Particularly \mathcal{G}_1 is the analytic functional class, and $\mathcal{G}_1 \subseteq \mathcal{G}_s (s \geq 1) \subseteq C^\infty$ and $C^\infty \subseteq \mathbb{F}^n[[x]]$ the set of n dimensional formal series. Now denote the resonant set by

$$\Lambda_r = \{(j, k, l) \mid \langle k, \mu \rangle = \mu_l, \quad j + |k| \geq 2, \quad k \in \mathbb{Z}_+^{n-1}, \quad j \in \mathbb{Z}_+, \quad l \in \{2, \dots, n\}\}.$$

Under the condition that the equilibrium $y = 0$ is not isolated, system (2) can be written in

$$\frac{dy_1}{dx} = \hat{f}_1(y_1, z), \quad \frac{dz}{dx} = A_0 z + \hat{f}_2(y_1, z), \quad (4)$$

with $z = (y_2, \dots, y_n)$, A_0 having the eigenvalues μ , and $\hat{f}_1(y_1, 0) \equiv 0$ and $\hat{f}_2(y_1, 0) \equiv 0$. Moreover, its Poincaré-Dulac formal normal form system is

$$\frac{dy_1}{dx} = 0, \quad \frac{dz}{dx} = A_0 z + g(y_1, z), \quad (5)$$

where $g(y_1, z) = \sum_{j,k,l \in \Lambda_r} g_{(j,k),l} y_1^j z^k e_l$ with e_l the l -th unit vector. Set

$$q = \min\{|k| \mid (j, k, l) \in \Lambda_r, \quad g_{(j,k),l} \neq 0, \quad \exists j, l\}, \quad (6)$$

and

$$q^* = \min\{j + |k| \mid (j, k, l) \in \Lambda_r, \quad g_{(j,k),l} \neq 0, \quad \exists l\}. \quad (7)$$

Formulating a function Φ as

$$c^{-1}\Phi(t) = \max\{|k \cdot \lambda|^{-1} \mid |k| \leq t, \quad k \in \mathbb{Z}_+^d\}, \quad (8)$$

with c a normalized parameter such that $\Phi(1) = 1$. Of course, when $\Phi(t) = t^\mu$, it is of the diophantine type. Next results from [25] characterize the (formal) Gevrey integrability via the diophantine type small divisor condition.

Theorem 8. *Assume that system (2) is of Gevrey- s smooth with the origin as a nonisolated equilibrium, and μ nonresonant. Then the following statements hold.*

- (a) *If A_0 has its eigenvalues all with positive real parts or all with negative real parts, then system (2) has a local Gevrey- s smooth first integral whose formal series is not trivial.*
- (b) *Assume that A is diagonal, and $\Phi(t) = t^\mu$ with $\mu > 0$ a constant.*
 - (b₁) *If $\partial_z \hat{f}_2(y_1, 0) \equiv 0$ in (4) and $q < \infty$ defined in (6), there exists a local Gevrey- s^* smooth first integrals with nontrivial formal series, where $s^* = \max\{s, (\mu + q)/(q - 1)\}$.*
 - (b₂) *If $q^* < \infty$ defined in (7), there exists a nontrivial formal Gevrey- s^* first integral, where $s^* = \max\{s - 1, (\mu + 1)/(q^* - 1)\}$.*

The proof will be done by using the homological equation, KAM theory, the Gevrey norm of functions and majorant Gevrey series together with lots of estimations.

After Theorem 8 we naturally have the next questions.

Open problem 3. Among analytic differential systems of form (1) satisfying that \mathcal{R}_0 is empty, and the origin is a nonisolated equilibrium and that μ have both elements with positive and negative real parts,

- (a) what is the subset in which all systems have analytic first integrals in a neighborhood of the origin?
- (b) what is the subset in which all systems have Gevrey first integrals in a neighborhood of the origin?

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