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Time-Reversibility and Ivariants of Some 3-dim Systems

Tatjana Petek and Valery G. Romanovski

Abstract. We study time-reversibility and invariants of the group of transformations $x \to x,\ y \to \alpha y,\ z \to \alpha^{-1} z$ for three-dimensional polynomial systems with $0 : 1 : -1$ resonant singular point at the origin. An algorithm to find the Zariski closure of the set of time-reversible systems in the space of parameters is proposed. The interconnection of time-reversibility and invariants of the group mentioned above is discussed.

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Dedicated to the memory of
Professor K. S. Sibirsky

1 Introduction

Let k be a field, let G be a multiplicative group of invertible $n \times n$ matrices with elements in k and, for $A \in G$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in k^n$, let $A \cdot \mathbf{x}$ denote the usual action of G on k^n. A polynomial $f \in k[x_1, \dots, x_n]$ is *invariant under* G if $f(\mathbf{x}) = f(A \cdot \mathbf{x})$ for every $\mathbf{x} \in k^n$ and every $A \in G$. The polynomial f is also called an *invariant* of G.

Consider two-dimensional systems of the form

$$\begin{aligned} \dot{x} &= x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q, \\ \dot{y} &= -y + \sum_{(p,q)\in S} b_{qp} x^q y^{p+1}, \end{aligned} \tag{1}$$

where the variables x and y and the coefficients of (1) are complex, and $S \subset (\{-1\} \cup \mathbb{N}_0) \times \mathbb{N}_0$ is a finite set, of which every element (p, q) satisfies $p + q \geq 1$. Let ℓ be the cardinality of the set S. Then, $\mathbb{C}^{2\ell}$ is the parameter space of (1), which we denote by $E(a, b)$. The set of polynomials in ordered variables $a_{p_1,q_1}, \dots, a_{p_\ell,q_\ell}, b_{q_\ell,p_\ell}, \dots, b_{q_1,p_1}$ with coefficients in the field k will be denoted by $k[a, b]$.

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After the transformation

$$x' = e^{-i\varphi}x, \quad y' = e^{i\varphi}y \tag{2}$$

(such transformations form a one-parametric group of the parameter φ), we obtain the system

$$\dot{x}' = x' - \sum_{(p,q)\in S} a(\varphi)_{pq} x'^{p+1} y'^{q}, \quad \dot{y}' = -y' + \sum_{(p,q)\in S} b(\varphi)_{qp} x'^{q} y'^{p+1},$$

where the coefficients of the transformed system are

$$a(\varphi)_{pq} = a_{pq} e^{i(p_j - q_j)\varphi}, \quad b(\varphi)_{qp} = b_{qp} e^{-i(p_j - q_j)\varphi}, \tag{3}$$

for $(p,q) \in S$. For any fixed φ the equations in (3) determine an invertible linear mapping U_φ of the space $E(a,b)$ of parameters of (1) onto itself.

The group U_φ of family (1) acts on $E(a,b) = \mathbb{C}^{2\ell}$. The set of polynomial invariants of this group action has been for the first time studied by Sibirsky [12, 13]. Actually, Sibirsky considered the case of the "real" system (1), that is, the case where both equations on the right-hand side of (1) are multiplied by i and the first equation of (1) is the complex conjugate of the second one (such systems are complexifications of real systems, see e.g. [9, Chapter 3]). However, as it is shown in [8] and [9, Chapter 5], the theory for general systems (1) is similar to the theory developed by Sibirsky.

Before we proceed, we fix some notations. For any n-tuple $s = (s_1, s_2, \ldots, s_n)$, $n \geq 1$, let $\hat{s}$ be the permutation $\hat{s} = (s_n, s_{n-1}, \ldots, s_1)$. For two n-tuples $r = (r_1, r_2, \ldots, r_n)$, $s = (s_1, s_2, \ldots, s_n)$ we define the "dot"-product as $r \cdot s = r_1 s_1 + r_2 s_2 + \cdots + r_n s_n$. Given n-tuples r, s, let the ordered pair (r, s) denote the $2n$-tuple generated in the obvious way. Furthermore, we will use a short form of monomial writing as $(a_1, a_2, \ldots, a_n)^{(\nu_1, \nu_2, \ldots, \nu_n)} := a_1^{\nu_1} a_2^{\nu_2} \ldots a_n^{\nu_n} = a^\nu$, where $a = (a_1, \ldots, a_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$.

Let $L_1, L_2 : \mathbb{N}_0^{2\ell} \to \mathbb{Z}$ be homomorphisms of the additive monoid $\mathbb{N}_0^{2\ell}$ defined with respect to the ordered set S by

$$\begin{aligned} L_1(\nu) &= p_1\nu_1 + \cdots + p_\ell \nu_\ell + q_\ell \nu_{\ell+1} + \cdots + q_1 \nu_{2\ell} \\ &= (p, \hat{q}) \cdot \nu, \\ L_2(\nu) &= q_1\nu_1 + \cdots + q_\ell \nu_\ell + p_\ell \nu_{\ell+1} + \cdots + p_1 \nu_{2\ell}) \\ &= (q, \hat{p}) \cdot \nu, \end{aligned} \tag{4}$$

where $p := (p_1, \ldots, p_\ell)$, $q := (q_1, \ldots, q_\ell)$ and $\nu := (\nu_1, \ldots, \nu_{2\ell})$. Furthermore, the map

$$L := L_1 - L_2 : \mathbb{N}_0^{2\ell} \to \mathbb{Z} \tag{5}$$

is a monoid-homomorphism as well, hence the kernel,

$$\widetilde{\mathcal{M}} := \ker L = \{\nu : \ L(\nu) = 0\} \tag{6}$$

is also a monoid. Since U_φ changes only the coefficients of polynomials, a polynomial $f \in \mathbb{C}[a,b]$ is an invariant of the group U_φ if and only if each of its terms is an invariant (see Lemma 3.4 of [12]). Therefore, for the description of polynomial invariants of U_φ, it suffices to find the invariant monomials. By (3), for $\nu \in \mathbb{N}_0^{2\ell}$, $a = (a_{p_1,q_1} \dots a_{p_\ell,q_\ell})$, $b = (b_{q_1,p_1} \dots a_{q_\ell,p_\ell})$, we denote by $[\nu] \in \mathbb{C}[a,b]$ the monomial

$$[\nu] := a_{p_1q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1p_1}^{\nu_{2\ell}} = (a,\hat{b})^\nu. \tag{7}$$

The image of ν under the group action U_φ is the monomial

$$\begin{aligned} U_\varphi([\nu]) &= (a(\varphi),\widehat{b(\varphi)})^\nu \\ &= a(\varphi)_{p_1q_1}^{\nu_1} \cdots a(\varphi)_{p_\ell q_\ell}^{\nu_\ell} b(\varphi)_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b(\varphi)_{q_1p_1}^{\nu_{2\ell}} \\ &= a_{p_1q_1}^{\nu_1} e^{i\varphi\nu_1(p_1-q_1)} \cdots a_{p_\ell q_\ell}^{\nu_\ell} e^{i\varphi\nu_\ell(p_\ell-q_\ell)} b_{q_\ell p_\ell}^{\nu_{\ell+1}} e^{i\varphi\nu_{\ell+1}(q_\ell-p_\ell)} \cdots b_{q_1p_1}^{\nu_{2\ell}} e^{i\varphi\nu_{2\ell}(q_1-p_1)} \\ &= e^{i\varphi[\nu_1(p_1-q_1)+\cdots+\nu_\ell(p_\ell-q_\ell)+\nu_{\ell+1}(q_\ell-p_\ell)+\cdots+\nu_{2\ell}(q_1-p_1)]} a_{p_1q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1p_1}^{\nu_{2\ell}} \\ &= e^{i\varphi(L_1-L_2)(\nu)}\,[\nu] \\ &= e^{i\varphi L(\nu)}\,[\nu]\,. \end{aligned} \tag{8}$$

From (8) we see that the monomial $[\nu]$ defined by (7) is invariant under the group action U_φ, for system (1) if and only if $L(\nu)=0$, that is, if and only if $\nu \in \widetilde{\mathcal{M}}$. Since, for any $\nu \in \mathbb{N}_0^{2\ell}$,

$$\begin{aligned} L(\nu) &= (p-q,\hat{q}-\hat{p}) \cdot \nu \\ &= (q-p,\hat{p}-\hat{q}) \cdot \hat{\nu} \\ &= -L(\hat{\nu}), \end{aligned} \tag{9}$$

we have $\nu \in \widetilde{\mathcal{M}}$ if and only if $\hat{\nu} \in \widetilde{\mathcal{M}}$, hence the monomial $[\nu]$ is invariant under the group action U_φ if and only if its so-called conjugate

$$\begin{aligned} [\hat{\nu}] &= a_{p_1q_1}^{\nu_{2\ell}} \cdots a_{p_\ell q_\ell}^{\nu_\ell+1} b_{q_\ell p_\ell}^{\nu_\ell} \cdots b_{q_1p_1}^{\nu_1} \\ &= (a,b)^{\hat{\nu}} \end{aligned} \tag{10}$$

is also invariant.

Sibirsky found some important properties of the monoid $\widetilde{\mathcal{M}}$. One of them is the fact that the set $\{[\nu] : \nu \in \widetilde{\mathcal{M}}\}$ is closed under multiplication. From his results one can see that a basis of the monoid $\widetilde{\mathcal{M}}$ (a basis of the invariants of the group U_φ) can be found by sorting, since Sibirsky got a bound for the degree of basis invariants. A simple algorithm to compute generators of $\widetilde{\mathcal{M}}$ based on the Gröbner bases theory was proposed in [4].

With system (1) and the monoind $\widetilde{\mathcal{M}}$ we associate the ideal

$$\widetilde{I}_S = \langle [\nu] - [\hat{\nu}] : \nu \in \widetilde{\mathcal{M}} \rangle.$$

This ideal was called in [4] the Sibirsky ideal of system (1). It was shown by Sibirsky [12, Chapter 3] that in the "real" case if the parameters of the system belong to the variety $\mathbf{V}(I_S)$, then the vector field of the system is symmetric with respect to a line passing through the origin (after reversion of time), that is, it is time-reversible, and, therefore, admits an analytic local first integral in a neighborhood of the origin. Later on the result was generalized to general systems (1) in [7,8], where it was shown that for family (1) not all systems from $\mathbf{V}(I_S)$ are time-reversible, but $\mathbf{V}(I_S)$ is the Zariski closure of the set of time-reversible systems and, therefore, all systems from $\mathbf{V}(I_S)$ admit an analytic first integral in a neighborhood of the origin.

We recall (see e.g. [5]) that in the higher-dimensional case a system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathcal{X}(\mathbf{x}), \tag{11}$$

where $\mathcal{X}(\mathbf{x})$ is a vector function defined on some domain D of $\mathbb{R}^n$ or $\mathbb{C}^n$, is *time-reversible* on D if there exists an involution $\psi : D \to D$ (the involution means that ψ is smooth and $\psi \circ \psi = id_D$) such that

$$D_\psi^{-1} \mathcal{X} \circ \psi = -\mathcal{X}.$$

It is said that a system (11) is *completely integrable* on D if it admits $n-1$ functionally independent analytic first integrals on D. The problem of complete integrability can be also considered as a natural generalization of the center problem for two-dimensional systems to higher dimensions, see e.g. [6, 11, 14].

In this paper we study three-dimensional systems of the form

$$\begin{aligned} \dot{x} &= P_1(x,y,z), \\ \dot{y} &= y + P_2(x,y,z), \\ \dot{z} &= -z + P_3(x,y,z), \end{aligned} \tag{12}$$

where P_j, $j \in \{1,2,3\}$, are polynomial functions on $\mathbb{C}^3$ which vanish together with its first partial derivatives at the origin and present some generalizations of the above mentioned results of Sibirsky and those of [7, 8] to the case of system (12).

2 Time-reversibility

The following statement is easily derived from a general result of [6] (see also [10]).

Theorem 1. *If under the interchange of the last two variables a system* (12) *is transformed to a system of the same form but with the right-hand side multiplied by* -1, *then it admits two analytic local first integrals of the form*

$$\Psi_1(x,y,z) = x + \cdots$$

and

$$\Psi_2(x,y,z) = yz + \cdots .$$

In the other words, the statement means that if a system (12) is time-reversible with respect to the linear involution defined on $\mathbb{C}^3$

$$x \mapsto x, \ y \mapsto z, \ z \mapsto y, \tag{13}$$

then it is completely integrable in a neighborhood of the origin.

Without loss of generality we can write a polynomial system (12) in the form

$$\begin{aligned} \dot{x} &= \sum_{(P,Q,R)\in T} a_{PQR} x^P y^Q z^R, \\ \dot{y} &= y - \sum_{(p,q,r)\in S} b_{pqr} x^p y^{q+1} z^r, \\ \dot{z} &= -z + \sum_{(p,q,r)\in S} c_{prq} x^p y^r z^{q+1}, \end{aligned} \tag{14}$$

where $S \subset \mathbb{N}_0 \times (\mathbb{N}_0 \cup \{-1\}) \times \mathbb{N}_0$ is a set of ℓ triplets, all satisfying $1 \le p+q+r \le N$, and $T \subset \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$ is a set of triplets, all satisfying $2 \le P+Q+R \le N$, where N is the degree of (14). Note that the indexing set T is symmetric with respect to the second and third coordinates, i.e. $(P,Q,R) \in T$ if and only if $(P,R,Q) \in T$.

The correctness of the following statement can be verified by straightforward computations, see also (20).

Lemma 2. *Let $\alpha \neq 0$. If a system* (14) *is time-reversible with respect to the involution*

$$\psi(x,y,z) = (x, \alpha z, \alpha^{-1} y), \tag{15}$$

then $a_{PQQ} = 0$ for every $(P,Q,Q) \in T$.

Due to the above lemma, we a priori assume that in (14)

$$a_{PQQ} = 0 \ \text{ for all } (P,Q,Q) \in T$$

or, equivalently, we exclude these parameters from the parameter space. By enumeration we fix an arbitrary order in the indexing set S

$$S = \{(p_1,q_1,r_1), \dots, (p_\ell,q_\ell,r_\ell)\}. \tag{16}$$

Further we split the indexing set T in a disjoint union $T = T_1 \cup T_2$ with $T_1 = \{(P,Q,R) : Q > R\}$ and $T_2 = \{(P,Q,R) : Q < R\}$. Note that T_1 and T_2 have the property that for every $(P,Q,R) \in T_1$ we have $(P,R,Q) \in T_2$, thus both T_1 and T_2 have the same number of elements, say m elements. Then we fix an arbitrary order in T_1:

$$T_1 = \{(P_1,Q_1,R_1), \dots, (P_m,Q_m,R_m)\}. \tag{17}$$

In a natural way, this order induces the order in the set T_2

$$T_2 = \{(P_1,R_1,Q_1), \dots, (P_m,R_m,Q_m)\}.$$

The ring of polynomials with ordered coefficients

$$a_{P_1Q_1R_1}, \cdots a_{P_mQ_mR_m}, a_{P_mR_mQ_m}, \cdots a_{P_1R_1Q_1}, b_{p_1q_1r_1}, \cdots b_{p_\ell q_\ell r_\ell}, c_{p_\ell r_\ell q_\ell}, \cdots c_{p_1r_1q_1} \tag{18}$$

as indeterminates and coefficients in a field k (typically $\mathbb{C}$ or $\mathbb{Q}$) will be denoted by $k[a,b,c]$. Along with the latter ring we will work also with its extension $k[a,b,c,\alpha,w]$ where α and w are variables.

Proposition 3. *1) The Zariski closure of the set of systems in family* (14) *which are time-reversible with respect to involution* (15) *is the variety* $\mathbf{V}(\mathcal{I}_R)$ *of the ideal*

$$\mathcal{I}_R = H \cap \mathbb{C}[a,b,c],$$

where H *is the following ideal in* $\mathbb{C}[a,b,c,\alpha,w]$

$$H = \langle a_{PQR}\alpha^Q + a_{PRQ}\alpha^R, b_{pqr}\alpha^{q+1} - c_{prq}\alpha^{r+1}, \alpha w - 1 : (P,Q,R) \in T,\ (p,q,r) \in S\rangle. \tag{19}$$

2) If the parameters of a system (14) *belong to the variety* $\mathbf{V}(\mathcal{I}_R)$*, then the system is completely integrable.*

Remark 4. *Notice that the above ideal* H *remains the same if we replace the indexing set* T *by only* T_1 *or by* T_2*.*

Proof of Prop. 3. Let $\mathcal{X}$ be the vector field (14). Equating to zero the coefficients of the monomials of the polynomial $D_\psi \cdot \mathcal{X} + \mathcal{X} \circ \psi$ we obtain the system

$$a_{PQR} = -\alpha^{R-Q} a_{PRQ},\ b_{pqr} = \alpha^{r-q} c_{prq}, \quad (P,Q,R) \in T,\ (p,q,r) \in S.$$

That means, system (14) is time-reversible with respect to involution (15) if and only if there is a nonzero α such that

$$a_{PQR}\alpha^Q + \alpha^R a_{PRQ} = 0, \quad b_{pqr}\alpha^q - \alpha^r c_{prq} = 0, \quad (P,Q,R) \in T,\ (p,q,r) \in S \tag{20}$$

or, equivalently, avoiding the possibly negative exponent $q \geq -1$

$$a_{PQR}\alpha^Q + \alpha^R a_{PRQ} = 0, \quad b_{pqr}\alpha^{q+1} - \alpha^{r+1} c_{prq} = 0, \quad (P,Q,R) \in T,\ (p,q,r) \in S.$$

By the Elimination theorem (see e.g. [2, 9]) this is the case when the coefficients of (14) belong to the variety of the ideal $\mathcal{I}_R$ defined by (3).

2) By the construction $\mathbf{V}(\mathcal{I}_R)$ is the Zariski closure of systems which are time-reversible with respect to (15). We observe that if a system (14) is time-reversible with respect to (15) then, after the change of coordinates $x_1 = x$, $x_2 = \alpha^{-1}y$, $x_3 = \alpha z$, we obtain the system which is time-reversible with respect to involution (13). By Theorem 1 the obtained system is completely integrable. Thus, $\mathbf{V}(\mathcal{I}_R)$ is the Zariski closure of a set of completely integrable systems. By the results of [11] the set of completely integrable systems is an algebraic set. Therefore systems from $\mathbf{V}(\mathcal{I}_R)$ are completely integrable. □

3 Invariants

Recalling the fixed order (18) in our polynomial indeterminates, we write each monomial in the polynomial ring with these coefficients as indeterminates in the form

$$a_{P_1Q_1R_1}^{\mu_1} \cdots a_{P_mQ_mR_m}^{\mu_n} a_{P_mR_mQ_m}{}^{\mu_{n+1}} \cdots a_{P_1R_1Q_1}^{\mu_{2m}} b_{p_1q_1r_1}^{\nu_1} \cdots b_{p_\ell q_\ell r_\ell}^{\nu_\ell} c_{p_\ell r_\ell q_\ell}^{\nu_{\ell+1}} \cdots c_{p_1r_1q_1}^{\nu_{2\ell}}. \tag{21}$$

Introducing the notations

$$a = (a_{P_1Q_1R_1}, \ldots, a_{P_mQ_mR_m}), \qquad b = (b_{p_1q_1r_1}, \ldots, b_{p_\ell q_\ell r_\ell}),$$
$$a' = (a_{P_1R_1Q_1}, \ldots, a_{P_mR_mQ_m}), \qquad c = (c_{p_1r_1q_1}, \ldots, c_{p_\ell r_\ell q_\ell}),$$

we set up the monomial (21)

$$\begin{aligned}[\mu; \nu] &= [\mu_1, \ldots, \mu_{2m}; \nu_1, \ldots, \nu_{2\ell}] \\ &= (a, \widehat{a'})^\mu (b, \hat{c})^\nu.\end{aligned} \tag{22}$$

In particular,

$$[\mu; 0] = (a, \widehat{a'})^\mu \tag{23}$$

and

$$[0; \nu] = (b, \hat{c})^\nu. \tag{24}$$

With systems (14) and the fixed enumeration (17), (16) of indices $(P, Q, R) \in T_1$ and $(p, q, r) \in S$ we associate vectors

$$\begin{aligned}K &= (Q_1 - R_1, \ldots, Q_m - R_m) = (K_1, \ldots, K_m), \\ \kappa &= (q_1 - r_1, \ldots, q_\ell - r_\ell) = (\kappa_1, \ldots, \kappa_\ell)\end{aligned}$$

and the map $L : \mathbb{N}_0^{2m} \times \mathbb{N}_0^{2\ell} \to \mathbb{Z}$, defined by

$$L(\mu, \nu) = (K, -\widehat{K}) \cdot \mu + (\kappa, -\hat{\kappa}) \cdot \nu, \qquad \mu \in \mathbb{N}_0^{2m},\ \nu \in \mathbb{N}_0^{2\ell}.$$

It is easy to see that L is a homomorphism of the Abelian monoid $\mathbb{N}_0^{2m} \times \mathbb{N}_0^{2\ell}$ into the Abelian monoid $\mathbb{Z}$ and consequently, the kernel of L, denoted by $\mathcal{M} := \{(\mu, \nu) : \ L(\mu, \nu) = 0\}$ is a submonoid in $\mathbb{N}_0^{2m} \times \mathbb{N}_0^{2\ell}$.

A simple computation gives that for every $\mu \in \mathbb{N}_0^{2m},\ \nu \in \mathbb{N}_0^{2\ell}$

$$L(\mu, \nu) = -L(\hat{\mu}, \hat{\nu}),$$

easily providing the following statement.

Lemma 5. $(\mu, \nu) \in \mathcal{M}$ *if and only if* $(\hat{\mu}, \hat{\nu}) \in \mathcal{M}$.

Let

$$x \to x, \quad y \to \alpha y, \quad z \to \alpha^{-1} z \tag{25}$$

be the one-parametric group U_α of invertible linear transformations of the phase space of systems (14). Similarly to the two-dimensional case in Section 1, we denote the coefficients of the new systems as $a_{PQR}(\alpha), b_{pqr}(\alpha), c_{prq}(\alpha)$. The straightforward computation gives

$$\begin{aligned} a_{PQR}(\alpha) &= \alpha^{R-Q} a_{PQR}, \\ b_{pqr}(\alpha) &= \alpha^{r-q} b_{pqr}, \\ c_{prq}(\alpha) &= \alpha^{q-r} c_{prq}, \end{aligned} \tag{26}$$

for all $(P, Q, R) \in T$, $(p, q, r) \in S$.

Proposition 6. *The monomial* $[\mu; \nu]$ *is invariant under the action of group* (25) *if and only if* $(\mu, \nu) \in \mathcal{M}$.

Proof. The action of the group (25) induces the change of coefficients of (14) according to (26). Recalling (23) and (24) and performing this substitution in $[\mu, \nu]$ we obtain

$$\begin{aligned} U_\alpha([\mu;\nu]) =& [\mu;\nu]\alpha^{(Q-R,\hat{R}-\hat{Q})\cdot\mu+(q-r,\hat{r}-\hat{q})\cdot\nu} \\ =& [\mu,\nu]\alpha^{(K,-\hat{K})\cdot\mu+(\kappa,-\hat{\kappa})\cdot\nu} \\ =& [\mu,\nu]\alpha^{L(\mu,\nu)} \end{aligned}$$

wherefrom the claim easily follows. □

We now define a generalized version of the Sibirsky ideal. For any $\mu \in \mathbb{N}_0^{2m}$ denote $|\mu| = \sum_{j=1}^{2m} \mu_j$.

Definition 7. *The ideal*

$$\mathcal{I}_S = \langle (-1)^{|\mu|}[\mu;\nu] - [\hat{\mu};\hat{\nu}] : (\mu,\nu) \in \widetilde{\mathcal{M}} \rangle$$

is called the Sibirsky ideal of systems (14).

For the proof of our main theorem, we will apply the following theorem ([1], Theorem 2.4.10).

Theorem 8. *Let* J *be an ideal of* $k[y_1, \dots, y_m]$, I *be an ideal of* $k[x_1, \dots x_n]$ *and let* $K = \langle I, y_1 - f_1, \dots, y_m - f_m \rangle \subseteq k[y_1, \dots, y_m, x_1, \dots x_n]$.
Let $\phi : k[y_1, \dots, y_m]/_J \to k[x_1, \dots x_n]/_I$ *be the homomorphism defined by*

$$y_i + J \mapsto f_i + I.$$

Then $\ker \phi = K \cap k[y_1, \dots, y_m] (\mathrm{mod} J)$. *That is, if* $\ker \phi = \langle g_1 + J, \dots, g_p + J \rangle$, *then* $K \cap k[y_1, \dots, y_m] = \langle g_1, \dots, g_p \rangle$.

The statement below is our main result and it generalizes a result obtained in [7] for the case of systems (1) to the case of systems (14).

Theorem 9. $\mathcal{I}_R = \mathcal{I}_S$.

Proof. Recall that the ideal H is defined by (19) and the ideal, which we are interested in, is $\mathcal{I}_R = H \cap \mathbb{C}(a,b,c)$. Let $\mathcal{I} = \langle \alpha w - 1\rangle$, $s = (s_1,\dots,s_m)$, $t = (t_1,\dots,t_\ell)$. We define a homomorphism $\phi : \mathbb{C}[a,b,c] \to \mathbb{C}[s,t,\alpha,w]_{/\mathcal{I}}$ by

$$\begin{aligned}
a_{P_nQ_nR_n} &\mapsto s_n + \mathcal{I},\\
a_{P_nR_nQ_n} &\mapsto -\alpha^{Q_n-R_n} s_n + \mathcal{I},\\
b_{p_jq_jr_j} &\mapsto t_j + \mathcal{I},\\
c_{p_jr_jq_j} &\mapsto \alpha^{q_j-r_j} t_j + \mathcal{I}, \quad \text{if } q_j \ge r_j,\\
c_{p_jr_jq_j} &\mapsto w^{r_j-q_j} t_j + \mathcal{I}, \quad \text{if } r_j > q_j,\\
& n = 1,2,\dots,m,\ j = 1,2,\dots,\ell.
\end{aligned}$$

Recalling the shorthand notation $K_n = Q_n - R_n > 0$, $n = 1,2,\dots,m$, and $\kappa_j = q_j - r_j$, $j = 1,2,\dots,\ell$, let

$$\begin{aligned}
\widetilde{H} = \langle \mathcal{I},\ & a_{P_nQ_nR_n} - s_n,\ a_{P_nR_nQ_n} - (-\alpha^{K_n} s_n),\ b_{p_jq_jr_j} - t_j,\ c_{p_{k_j}r_{k_j}q_{k_j}} - t_{k_j}\alpha^{\kappa_{k_j}},\\
& c_{p_{k_i}r_{k_i}q_{k_i}} - w^{-\kappa_{k_i}} t_{k_i} : 1 \le n \le m, 1 \le j \le \ell, \kappa_{k_j} \ge 0, \kappa_{k_i} < 0\rangle.
\end{aligned}$$

By Theorem 8 (J is taken to be trivial), we have

$$\ker\phi = \widetilde{H} \cap \mathbb{C}[a,b,c]$$

and by Proposition 3, $\mathcal{I}_R = H \cap \mathbb{C}[a,b,c]$.

We next show that $\widetilde{H} \cap \mathbb{C}[a,b,c] = H \cap \mathbb{C}[a,b,c]$. By elimination of $s_1,\dots,s_m,t_1,\dots,t_\ell$ from $\widetilde{H}$ we get exactly H. Hence $H = \widetilde{H} \cap \mathbb{C}[a,b,c,\alpha,w]$ and

$$\begin{aligned}
\mathcal{I}_R &= H \cap \mathbb{C}[a,b,c]\\
&= \widetilde{H} \cap \mathbb{C}[a,b,c,\alpha,w] \cap \mathbb{C}[a,b,c]\\
&= \widetilde{H} \cap \mathbb{C}[a,b,c]\\
&= \ker\phi.
\end{aligned}$$

Next we check that $\mathcal{I}_S \subset \ker\phi$, i.e. that

$$\phi([\hat\mu;\hat\nu]) = (-1)^{|\mu|}\phi([\mu;\nu]), \qquad (\mu;\nu) \in \mathcal{M}.$$

Writing in a short way, with $\mu = (\xi,\eta) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$, $\nu = (\zeta,\theta) \in \mathbb{N}_0^\ell \times \mathbb{N}_0^\ell$, we have

$$[\mu;0] = [\xi,\eta;0] = a^\xi(\hat{a}')^\eta = \prod_{j=1}^m a_{P_jQ_jR_j}{}^{\xi_j} \prod_{n=1}^m a_{P_nR_nQ_n}{}^{\hat\eta_n}$$

and

$$[0;\nu] = [0;\zeta,\theta] = b^\zeta \hat{c}^\theta = \prod_{j=1}^\ell b_{p_jq_jr_j}{}^{\zeta_j} \Pi_{n=1}^\ell c_{p_nr_nq_n}{}^{\hat\theta_n}.$$

Now, acting by ϕ on $[\mu;0]$, noting that $\hat{s}^\eta = s^{\hat{\eta}}$ gives us

$$\begin{aligned}\phi([\mu;0]) = \phi([\xi,\eta;0]) &= (-1)^{|\eta|}\alpha^{(\widehat{Q}-\widehat{R})\cdot\eta}s^\xi\hat{s}^\eta + \mathcal{I}\\ &= (-1)^{|\eta|}\alpha^{\widehat{K}\cdot\eta}s^{\xi+\hat{\eta}} + \mathcal{I}\end{aligned} \tag{27}$$

and

$$\begin{aligned}\phi([\hat{\mu};0]) = \phi([\hat{\eta},\hat{\xi};0]) &= (-1)^{|\xi|}\alpha^{(\widehat{Q}-\widehat{R})\cdot\hat{\xi}}s^{\hat{\eta}}\hat{s}^{\hat{\xi}} + \mathcal{I}\\ &= (-1)^{|\xi|}\alpha^{(Q-R)\cdot\xi}s^{\xi+\hat{\eta}} + \mathcal{I}\\ &= (-1)^{|\xi|}\alpha^{K\cdot\xi}s^{\xi+\hat{\eta}} + \mathcal{I}.\end{aligned}$$

By choosing $[\mu;0] = [\xi,\eta;0] \in \mathcal{M}$ we know that $K\cdot\xi = \widehat{K}\cdot\eta$. Moreover, it is easy to check that

$$\phi((-1)^{|\mu|}\,[\mu;0]) = \phi([\hat{\mu};0]))$$

since $(-1)^{|\xi|+2|\eta|} = (-1)^{|\xi|}$. We have to be a bit careful when computing $\phi(c^\theta)$. Namely, $\phi(c_n^{\theta_n}) = t_n^{\theta_n}\alpha^{\kappa_n\theta_n}$ if $\kappa_n = q_n - r_n \geq 0$ and $\phi(c_j^{\theta_j}) = t^j w^{-\kappa_j\theta_j}$ if $\kappa_j < 0$. Denote by κ_+ the non-negative part of κ, and by κ_- the negative part such that $\kappa = \kappa_+ + \kappa_-$ and $\operatorname{supp}\kappa_+ \cap \operatorname{supp}\kappa_- = \{\}$. Now,

$$\phi([0;\nu]) = \phi([0;\zeta,\theta]) = t^\zeta\hat{t}^\theta\alpha^{(\kappa_+)\cdot\hat{\theta}}w^{-(\kappa_-)\cdot\hat{\theta}} + \mathcal{I} = t^{\zeta+\hat{\theta}}\alpha^{(\kappa_+)\cdot\hat{\theta}}w^{-(\kappa_-)\cdot\hat{\theta}} + \mathcal{I} \tag{28}$$

and

$$\phi([0;\hat{\nu}]) = \phi([0;\hat{\theta},\hat{\zeta}]) = t^{\hat{\theta}+\zeta}\alpha^{(\kappa_+)\cdot\zeta}w^{-(\kappa_-)\cdot\zeta} + \mathcal{I}.$$

We next show that $\alpha^{(\kappa_+)\cdot\hat{\theta}}w^{-(\kappa_-)\cdot\hat{\theta}} - \alpha^{(\kappa_+)\cdot\zeta}w^{-(\kappa_-)\cdot\zeta} \in \mathcal{I}$ as soon as $(0;\zeta,\theta) \in \mathcal{M}$. Denote $u_1 = \kappa_+\cdot\hat{\theta}$, $u_2 = -\kappa_-\cdot\hat{\theta}$, $v_1 = \kappa_+\cdot\zeta$, $v_2 = -\kappa_-\cdot\zeta$. The requirement $(0;\zeta,\theta) \in \mathcal{M}$ tells us that $v_1 - u_1 = v_2 - u_2 =: d$. Assuming that $d \geq 0$ we obtain

$$\begin{aligned}\alpha^{(\kappa_+)\cdot\hat{\theta}}w^{-(\kappa_-)\cdot\hat{\theta}} - \alpha^{(\kappa_+)\cdot\zeta}w^{-(\kappa_-)\cdot\zeta} =& \alpha^{u_1}w^{u_2} - \alpha^{v_1}w^{v_2}\\ =& \alpha^{u_1}w^{u_2}(1-\alpha^d w^d)\\ =& \alpha^{u_1}w^{u_2}f(\alpha,w)(1-\alpha w)\end{aligned}$$

where $f(\alpha,w)$ is a polynomial. We proceed very similarly when $d < 0$. Therefore, $\phi([0;\nu]) = \phi([0;\hat{\nu}])$.

To complete this step of the proof, i.e. to show that all generating binomials of $\mathcal{I}_S$ are in the kernel of ϕ, let $(\mu,\nu) \in \mathcal{M}$. Then, as ϕ is a ring homomorphism,

$$\begin{aligned}\phi((-1)^{|\mu|}[\mu;\nu]) =& \phi((-1)^{|\mu|}[\mu;0])\phi([0;\nu])\\ =& \phi([\hat{\mu};0])\phi([0;\hat{\nu}])\\ =& \phi([\hat{\mu};\hat{\nu}]).\end{aligned}$$

It remains to check that $\ker\phi \subset I_S$. A reduced Gröbner basis G of $\mathbb{C}[a,b,c] \cap \widetilde{H}$ can be found by computing a reduced Gröbner basis of $\widetilde{H}$ using an elimination ordering with $\{a,b,c\} < \{w,\alpha,s,t\}$, and then intersecting it with $\mathbb{C}[a,b,c]$. Since $\widetilde{H}$

is binomial, any reduced Gröbner basis G of $\widetilde{H}$ also consists of binomials. This means that $\mathcal{I}_R = \widetilde{H} \cap \mathbb{Q}[a, b, c] = \ker \phi$ is a binomial ideal. Assume that for some $(\xi, \eta; \zeta, \theta)$, $(\gamma, \delta; \varepsilon, \varphi) \in \mathbb{N}^{2m} \times \mathbb{N}^{2\ell}$, $u \in \mathbb{C}$, the equality $\phi(u[\xi, \eta; \zeta, \theta] - [\gamma, \delta; \varepsilon, \varphi]) = 0$ holds. Without loosing any generality, we assume that $[\xi, \eta; \zeta, \theta]$ and $[\gamma, \delta; \varepsilon, \varphi]$ do not have nontrivial common factors. This implies that $\xi_j \gamma_j = \eta_j \delta_j = 0$, $j = 1, 2, \ldots, m$, and $\zeta_i \varepsilon_i = \theta_i \delta_i = 0$, $i = 1, 2, \ldots, \ell$. Suppose

$$\phi(u[\xi, \eta; \zeta, \theta]) = \phi([\gamma, \delta; \varepsilon, \varphi]).$$

We will show that $[\gamma, \delta; \varepsilon, \varphi] = [\hat{\eta}, \hat{\xi}; \hat{\theta}, \hat{\zeta}]$ and $u = (-1)^{|\xi|+|\eta|}$. From (27) and (28) one derives that

$$f := u(-1)^{|\eta|+|\delta|} s^{\xi+\hat{\eta}} t^{\zeta+\hat{\theta}} \alpha^{\widehat{K}\cdot\eta+(\kappa_+)\cdot\hat{\theta}} w^{-(\kappa_-)\cdot\hat{\theta}} - s^{\gamma+\hat{\delta}} t^{\varepsilon+\hat{\varphi}} \alpha^{\widehat{K}\cdot\delta+(\kappa_+)\cdot\hat{\varphi}} w^{-(\kappa_-)\cdot\hat{\varphi}} \in \langle \alpha w - 1 \rangle.$$

Computing the value of f at $w = \alpha^{-1}$ we must have 0. But this implies the equality of (possibly rational) monomials

$$s^{\xi+\hat{\eta}} t^{\zeta+\hat{\theta}} \alpha^{\widehat{K}\cdot\eta+(\kappa_+ + \kappa_-)\cdot\hat{\theta}} = s^{\gamma+\hat{\delta}} t^{\varepsilon+\hat{\varphi}} \alpha^{\widehat{K}\cdot\delta+(\kappa_+ + \kappa_-)\cdot\hat{\varphi}} \tag{29}$$

and additionally,

$$u(-1)^{|\eta|+|\delta|} = 1. \tag{30}$$

Comparing the powers at s, t, α in (29) gives

$$\xi + \hat{\eta} = \gamma + \hat{\delta} \tag{31}$$

$$\zeta + \hat{\theta} = \varepsilon + \hat{\varphi} \tag{32}$$

$$\widehat{K} \cdot \eta + \kappa \cdot \hat{\theta} = \widehat{K} \cdot \delta + \kappa \cdot \hat{\varphi}. \tag{33}$$

We will firstly prove and then immediately apply the following technical lemma.

Lemma 10. *Let $\xi, \eta, \gamma, \delta \in \mathbb{N}_0$ be non-negative integers. Assume that*

$$\xi + \eta = \gamma + \delta \tag{34}$$

$$\xi\gamma = 0 \tag{35}$$

$$\eta\delta = 0. \tag{36}$$

Then $(\gamma, \delta) = (\eta, \xi)$.

Proof. Let us firstly assume that $\gamma > \eta$. Then $\gamma \neq 0$ and by (35), $\xi = 0$. Apply (34) to get a contradiction, since δ is not negative. Similarly, if $\eta > \gamma$ we have $\eta \neq 0$ and thus by (36) one obtains $\delta = 0$. This contradicts the non-negativity of ξ. It follows that $\gamma = \eta$ and consequently from (34), $\delta = \xi$ as claimed. □

Let us continue with the proof of Theorem 9. From (31) we observe that $\xi_j + \hat{\eta}_j = \gamma_j + \hat{\delta}_j$ and by our assumption on coprimeness, $\xi_j \gamma_j = \hat{\eta}_j \hat{\delta}_j = 0$ for all $j = 1, 2, \ldots, m$. Applying Lemma 10 we obtain $\gamma_j = \hat{\eta}_j$ and $\hat{\delta}_j = \xi_j$, $j = 1, \ldots, m$

and in turn, $\gamma = \hat{\eta}$ and $\delta = \hat{\xi}$, i.e. $(\gamma, \delta) = (\hat{\eta}, \hat{\xi})$. In a very similar manner we get $(\varepsilon, \varphi) = (\hat{\theta}, \hat{\zeta})$ from (32).

It remains to see that both $[\xi, \eta; \zeta, \theta]$ and $[\gamma, \delta; \varepsilon, \varphi]$ must be in $\mathcal{M}$. By Lemma 5 and inserting $(\gamma, \delta, \varepsilon, \varphi) = (\hat{\eta}, \hat{\xi}, \hat{\theta}, \hat{\zeta})$ into (33) we confirm the claim.

Finally we easily get $u = (-1)^{|\xi|+|\eta|}$ from (30) since $|\delta| = |\hat{\xi}| = |\xi|$. □

A generating set or basis $\mathcal{N}$ of $\mathcal{M}$ is *minimal* if, for each $\nu \in \mathcal{N}$, $\mathcal{N} \setminus \{\nu\}$ is not a generating set. A minimal generating set is called a *Hilbert basis* of $\mathcal{M}$.

Theorem 11. *Let G be the reduced Gröbner basis of $\mathcal{I}_S$ with respect to a chosen term order. Then the following holds.*

1. *Every element of G has the form $(-1)^{|\mu|}[\mu; \nu] - [\hat{\mu}; \hat{\nu}]$, where $(\mu, \nu) \in \mathcal{M}$ and $[\mu; \nu]$ and $[\hat{\mu}; \hat{\nu}]$ have no common factors.*

2. *The set*

$$\begin{aligned}\mathcal{N} = \{(\mu, \nu), (\hat{\mu}, \hat{\nu}) : (-1)^{|\mu|}[\mu; \nu] - [\hat{\mu}; \hat{\nu}] \in G\} \\ \cup \{(0, e_j) + (0, \ e_{2\ell-j+1}) : j = 1, \dots, \ell \text{ and } \pm([0; e_j] - [0; e_{2\ell-j+1}]) \notin G\},\end{aligned}$$

where $e_j = (0, \dots, 0, \overset{j}{1}, 0, \dots, 0) \in \mathbb{Q}^{2\ell}$, is a Hilbert basis of $\mathcal{M}$.

The proof of the theorem is similar to the proof of Theorem 5.2.5 in [9].

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Tatjana Petek
Faculty of Electrical Engineering and Computer Science,
University of Maribor, Koroška cesta 46, SI-2000 Maribor,
Slovenia
Institute of Mathematics, Physics and Mechanics,
Jadranska 19, SI-1000 Ljubljana, Slovenia
E-mail: *tatjana.petek@um.si*

Valery G. Romanovski
Faculty of Electrical Engineering and Computer Science,
University of Maribor, Koroška cesta 46, SI-2000 Maribor,
Slovenia
Center for Applied Mathematics and Theoretical Physics,
Mladinska 3, SI-2000 Maribor, Slovenia
Faculty of Natural Science and Mathematics, University of
Maribor, Koroška cesta 160, SI-2000 Maribor, Slovenia
E-mail: *valerij.romanovskij@um.si*