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Properties of coverings in lattices of ring topologies

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Abstract. When studying unrefinable chains of ring topologies, it is natural to find out how neighborhoods of zero of ring topologies in such chains are related to each other.
It is proved that for any ideal the restrictions of these topologies to the ideal coincides, or the sum of any neighborhood of zero in the stronger topology with the intersection of the ideal with any neighborhood of zero in the weaker topology is a neighborhood of zero in the weaker topology. We construct a ring and two ring topologies which form an unrefinable chain in the lattice of all ring topologies that a basis of filter of neighborhoods of zero which consists of subgroups of the additive group of the ring and restriction of these topologies to some ideal of the ring is no longer a unrefinable chain. This example shows that the given in [4] conditions under which the properties of a unrefinable chain of ring topologies, are preserved under taking the supremum are essential.

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1 Introduction

This work is a continuation of the articles $[2-4]$ and is devoted to the study of the properties of finite unrefinable chains of ring topologies.

The main result is Theorem 3.2, which proves that if Δ is the lattice of all ring topologies on the ring R or the lattice of all ring topologies in each of which the topological ring has a basis of filter of neighborhoods of zero which consists of subgroups of additive group of the ring R, then for any ring topologies τ_1 and τ_2 of the ring R such that $\tau_1 \prec_\Delta \tau_2$, and for any ideal I of the ring R either $sup\{\tau_1 \tau(I)\} = sup\{\tau_2, \tau(I)\}$ or for any bases of filters of neighborhoods of zero $\{U_\gamma | \gamma \in \Gamma\}$ and $\{V_\beta | \beta \in \mathrm{B}\}$ in the topological rings (R, τ_1) and (R, τ_2), respectively, the family $\{(U_\gamma \bigcap I) + V_\beta | \gamma \in \Gamma, \beta \in \mathrm{B}\}$ will be a basis of filter of neighborhoods of zero in the topological ring (R, τ_1).

Previously (see [4]) it was proved that if Δ is the lattice of all ring topologies on the ring R or the lattice of all ring topologies in each of which the topological ring has a basis of filter of neighborhoods of zero which consists of subgroups of additive group of a ring, and if τ_1 and τ_2 are ring topologies on R such that $\tau_1 \prec_\Delta$

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τ_2 and $sup\{\tau_1, \tau(R^n)\} = sup\{\tau_2, \tau(R^n)\}$ for some natural number n, then either $sup\{\tau, \tau_1\} = sup\{\tau, \tau_2\}$ or $sup\{\tau, \tau_1\} \prec_\Delta sup\{\tau, \tau_2\}$ for any ring topology $\tau \in \Delta$, and in particular, $sup\{\tau(I), \tau_1\} \prec_\Delta sup\{\tau(I), \tau_2\}$ for any ideal I of the ring R.

We constructed an example showing that for the correctness of the previous result of the equality $sup\{\tau_1, \tau(R^n) = sup\{\tau_2, \tau(R^n)\}$ for some natural number n is essential . Namely, we construct a ring R such that for the lattice Δ of all ring topologies on the ring R in each of which the topological ring has a basis of filter of neighborhoods of zero which consists of subgroups of the additive group of the ring, there exist an ideal I and two ring topologies τ_1 and τ_2 such that $\tau_1 \prec_\Delta \tau_2$ and between topologies $sup\{\tau(I), \tau_1\}$ and $sup\{\tau(I), \tau_2\}$ there are an infinite number of ring topologies.

2 Notations and preliminaries

2.1. $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ is the field of real numbers;

2.2. $\mathfrak{N}$ is a free ultrafilter on the set $\mathbb{N}$;

2.3. Consider the associative ring $R = \mathbb{R} \rightthreetimes \mathbb{R}$, which is a semidirect product of the rings $\mathbb{R}$ and $\mathbb{R}$ (see [1]), i.e. on the set $R = \{(q, q')|q, q' \in \mathbb{R}\}$ the following operations are defined: $(q_1, q_1') + (q_2, q_2') = (q_1 + q_2, q_1' + q_2')$ and $(q_1, q_1') \cdot (q_2, q_2') = (q_1 \cdot q_2 + q_1 \cdot q_2' + q_1' \cdot q_2, q_1' \cdot q_2')$;

2.4. For each natural number i consider the rings $R_i = R$ and $R_i' = \{(0, r)|(0, r) \in R_i\}$;

2.5. $\widetilde{R}$ is the direct sum $\sum_{i=1}^{\infty} R_i$ and for any $i \in \mathbb{N}$ consider the canonical projection $pr_i : \widetilde{R} \to R_i$ (below we will identify the rings R_k and $\{\widetilde{r} \in \widetilde{R}|pr_i(\widetilde{r}) = (0, 0)$ for $i \neq k\}$);

2.6. For any set $A \in \mathfrak{N}$ consider the following sets:
$\widetilde{W_A} = \{\widetilde{r} \in \widetilde{R}|pr_i(\widetilde{r}) = (0, 0)$ for $i \notin A\}$, i.e. $\widetilde{W_A} = \sum_{i \in A} R_i$;
$\widetilde{W_A'} = \{\widetilde{r} \in \widetilde{W_A}|pr_i(\widetilde{r}) = (0, q)$ for $i \in A\}$, i.e. $\widetilde{W_A'} = \sum_{i \in A} R_i'$;

2.7. For any subring M of the ring $\mathbb{R}$ and any set $A \in \mathfrak{N}$, consider the set $\widetilde{W_{A,M}} = \{\widetilde{r} \in \widetilde{W_A}|pr_i(\widetilde{r}) = (q, 0)$ for $i \in A$, where $q \in M\}$.

Theorem 2.8. (see [1], Proposition 1.2.1 and Proposition 1.2.2). If Q is an arbitrary ring, then a family Ω of subsets of the ring Q is a basis of filter of neighborhoods of zero for some ring topology τ on the ring Q if and only if the family Ω satisfies the following conditions:

1. $0 \in \bigcap_{W \in \Omega} W$;
2. $\forall W_1, W_2 \in \Omega\ \exists W_3 \in \Omega$ such that $W_3 \subseteq W_1 \bigcap W_2$;
3. $\forall W_1 \in \Omega\ \exists W_2 \in \Omega$ such that $W_2 + W_2 \subseteq W_1$;
4. $\forall W_1 \in \Omega\ \exists W_2 \in \Omega$ such that $-W_2 \subseteq W_1$;

5. $\forall W_1 \in \Omega\ \exists W_2 \in \Omega$ such that $W_2 \cdot W_2 \subseteq W_1$
6. $\forall W_1 \in \Omega$ and $\forall a \in Q\ \exists W_2 \in \Omega$ such that $a \cdot W_2 \subseteq W_1$ and $W_2 \cdot a \subseteq W_1$.

Theorem 2.9. If Q is an arbitrary ring, τ_1 and τ_2 are ring topologies on Q, and $\{U_\gamma | \gamma \in \Gamma\}$ and $\{V_\beta | \beta \in \mathrm{B}\}$ are bases of filters of neighborhoods in the topological rings (Q, τ_1) and (Q, τ_2), respectively, then $\{V_\beta \bigcap U_\gamma | \beta \in \mathrm{B}, \gamma \in \Gamma\}$ will be the basis of filter of neighborhoods of zero in the topological ring $(Q, \sup\{\tau_1, \tau_2\})$.

Remark 2.10. If Q is an arbitrary ring and I is an ideal of the ring Q, then the family $\{I\}$ satisfies conditions $1 - 6$ of Theorem 2.8, and hence the family $\{I\}$ is a basis of filter of neighborhoods of zero for some ring topology on the ring Q, which will be denoted by $\tau(I)$.

Definition 2.11. If a and b are elements of a partially ordered set $(M, <)$ such that $a < b$ and $\{x \in M | a < x < b\} = \emptyset$, then, as usual, we say that the element b covers the element a, and we will write $a \prec_M b$.

3 Basic results

Theorem 3.1. *Let Q be an arbitrary ring and let Δ be the lattice of all ring topologies on the ring Q or the lattice of all ring topologies of the ring Q in each of which the topological ring has a basis of filter of neighborhoods of zero which consists of subgroups of the additive group of the ring Q. If τ_1 and τ_2 are ring topologies such that $\tau_1 \prec_\Delta \tau_2$ and $\{U_\gamma | \gamma \in \Gamma\}$ and $\{V_\beta | \beta \in \mathrm{B}\}$ are bases of filters of neighborhoods of zero in topological rings (Q, τ_1) and (Q, τ_2), respectively, then for any ideal I of ring Q there exist ring topologies $\bar{\tau}_1$ and $\bar{\tau}_2$ such that families $\{I + U_\gamma | \gamma \in \Gamma\}$ and $\{I + V_\beta | \beta \in \mathrm{B}\}$ are bases of filters of neighborhoods of zero in the topological rings $(Q, \bar{\tau}_1)$ and $(Q, \bar{\tau}_2)$ respectively, with $\bar{\tau}_1 \prec_\Delta \bar{\tau}_2$ or $\bar{\tau}_1 = \bar{\tau}_2$.*

Proof. From the fact that each of families $\{U_\gamma | \gamma \in \Gamma\}$ and $\{V_\beta | \beta \in \mathrm{B}\}$ satisfies the conditions of Theorem 2.8 and the fact that I is an ideal of the ring R, it easily follows that each of these families $\{I + U_\gamma | \gamma \in \Gamma\}$ and $\{I + V_\beta | \beta \in \mathrm{B}\}$ satisfies the conditions of Theorem 2.8 and hence each of them is a basis of filter of neighborhoods of zero for some ring topologies $\bar{\tau}_1$ and $\bar{\tau}_2$, respectively, of the ring Q.

Since $\tau_1 < \tau_2$, then for any element $\gamma \in \Gamma$ there exists an element $\beta \in \mathrm{B}$ such that $V_\beta \subseteq U_\gamma$. Then $I + V_\beta \subseteq I + U_\gamma$ and hence $\bar{\tau}_1 \le \bar{\tau}_2$.

To complete the proof of the theorem, it remains to check that $\bar{\tau}_1 \prec_\Delta \bar{\tau}_2$ or $\bar{\tau}_1 = \bar{\tau}_2$.

Assume the contrary, i.e. that $\bar{\tau}_1 < \bar{\tau} < \bar{\tau}_2$ for some ring topology $\bar{\tau}$ of the ring Q, and let $\{W_\delta | \delta \in \Lambda\}$ be a basis of filter of neighborhoods of zero of the topological ring $(Q, \bar{\tau})$.

Let us first verify that the family $\{I + W_\delta | \delta \in \Lambda\}$ is also a basis of filter of neighborhoods of zero of the topological ring $(Q, \bar{\tau})$.

Since $W_\delta \subseteq I + W_\delta$, then the set $I + W_\delta$ is a neighborhood of zero in the topological ring $(Q, \bar{\tau})$ for any $\delta \in \Lambda$. Moreover, for any $\delta_1 \in \Lambda$ there exists $\delta_2 \in \Lambda$ such that $W_{\delta_2} + W_{\delta_2} \subseteq W_{\delta_1}$. Since $\bar{\tau} < \bar{\tau}_2$, then there exists an element $\gamma \in \Gamma$ such that $V_\gamma + I \subseteq W_{\delta_2}$. Then $W_{\delta_2} + I \subseteq W_{\delta_2} + (V_\gamma + I) \subseteq W_{\delta_2} + W_{\delta_2} \subseteq W_{\delta_1}$. From the arbitrariness of $\delta_1 \in \Lambda$ it follows that the family $\{I + W_\delta | \delta \in \Lambda\}$ is a basis of filter of neighborhoods of zero in the topological ring $(Q, \bar{\tau})$.

Since $\tau_1 \prec_\Delta \tau_2$ and $\bar{\tau} \leq \bar{\tau}_2 \leq \tau_2$, then $\tau_1 \leq \sup\{\bar{\tau}, \tau_1\} \leq \tau_2$, and so $\tau_1 = \sup\{\bar{\tau}, \tau_1\}$ or $\sup\{\bar{\tau}, \tau_1\} = \tau_2$.

If $\tau_1 = \sup\{\bar{\tau}, \tau_1\}$ and $I + W_{\delta_0} \in \{I + W_\delta | \delta \in \Lambda\}$ then $\bar{\tau} \leq \tau_1$, and hence there exists $U_{\gamma_0} \in \{U_\gamma | \gamma \in \Gamma\}$ such that $U_{\gamma_0} \subseteq I + W_{\delta_0}$. Then $I + U_{\gamma_0} \subseteq I + W_{\delta_0}$, and so the set $I + W_{\delta_0}$ is a neighborhood of zero in the topological ring $(Q, \bar{\tau}_1)$.

From the arbitrariness of the neighborhood of zero $I + W_{\delta_0}$ in the topological ring $(Q, \bar{\tau})$ it follows that $\bar{\tau}_1 \geq \bar{\tau}$, and hence $\bar{\tau}_1 = \bar{\tau}$. We have obtained a contradiction with the assumption that $\bar{\tau}_1 < \bar{\tau}$ and therefore the case $\tau_1 = \sup\{\bar{\tau}, \tau_1\}$ is not possible.

Now let $\sup\{\bar{\tau}, \tau_1\} = \bar{\tau}_2$ and let $I + V_{\beta_0} \in \{I + V_\beta | \beta \in \mathrm{B}\}$. Then there exist $U_{\gamma_1} \in \{U_\gamma | \gamma \in \Gamma\}$ and $I + W_{\delta_1} \in \{I + W_\delta | \delta \in \Lambda\}$ such that $U_{\gamma_1} \bigcap (I + W_{\delta_1}) \subseteq I + V_{\beta_0}$.

Since $(I + U_{\gamma_1}) \bigcap (I + W_{\delta_1}) = I + U_{\gamma_1} \bigcap (I + W_{\delta_1}) \subseteq I + V_{\beta_0}$, then $I + V_{\beta_0}$ is a neighborhood of zero in the topological ring $(Q,\ sup\{\bar{\tau}_1, \bar{\tau}\})$. Since the set $I + V_{\beta_0} \in \{I + V_\beta | \beta \in \mathrm{B}\}$ is arbitrary, then it follows that $\bar{\tau}_2 \leq \sup\{\bar{\tau}_1, \bar{\tau}\}$ and since $\bar{\tau}_1 < \bar{\tau}$ then $\bar{\tau}_2 \leq \sup\{\bar{\tau}_1, \bar{\tau}\} = \bar{\tau}$. We got a contradiction with the fact that $\bar{\tau} < \bar{\tau}_2$, and hence the case $\sup\{\bar{\tau}, \tau_1\} = \tau_2$ is also impossible.

Therefore, the assumption that $\bar{\tau}_1 < \bar{\tau} < \bar{\tau}_2$ is not true.

This completes the proof of the theorem.

Theorem 3.2. *Let Q be an arbitrary ring and let Δ be the lattice of all ring topologies of the ring Q or the lattice of all ring topologies in each of which the topological ring has a basis of filter of neighborhoods of zero which consists of subgroups of the additive group of the ring Q. If τ_1 and τ_2 are ring topologies on Q such that $\tau_1 \prec_\Delta \tau_2$ and if $\{U_\gamma | \gamma \in \Gamma\}$ and $\{V_\beta | \beta \in \mathrm{B}\}$ are bases of filters of neighborhoods of zero in topological rings (Q, τ_1) and (Q, τ_2), respectively, then for any ideal I of ring Q either $\sup\{\tau(I), \tau_1\} = \sup\{\tau(I), \tau_2\}$ or the set $\{(U_\gamma \bigcap I) + V_\beta | \gamma \in \Gamma, \beta \in \mathrm{B}\}$ will be a basis of the filter of neighborhoods of zero in the topological ring (Q, τ_1).*

Proof. Let us first verify that the family $\Omega = \{V_\beta + (U_\gamma \bigcap I) | \beta \in B, \gamma \in \Gamma\}$ is the basis of filter of neighborhoods of zero in the topological ring (Q, τ) for some ring topology τ.

The fulfillment of conditions 1 – 4 and 6 of Theorem 2.8 follows from the fact that these conditions are satisfied for each of families $\{U_\gamma | \gamma \in \Gamma\}$ and $\{V_\beta | \beta \in B\}$ and the fact that I is an ideal of the ring Q.

Let us check that the family Ω also satisfies condition 5.

Let $V_{\beta_1} + (U_{\gamma_1} \bigcap I) \in \Omega$. Since families $\{U_\gamma | \gamma \in \Gamma\}$ and $\{V_\beta | \beta \in B\}$ are the bases of filters of neighborhoods of zero in the topological rings (R, τ_1) and (R, τ_2)

respectively, then there exists an element $\beta_2 \in \mathrm{B}$ such that $V_{\beta_2} \cdot V_{\beta_2} \subseteq V_{\beta_1}$, and there exist elements $\gamma_2, \gamma_3 \in \Gamma$ such that $U_{\gamma_2} + U_{\gamma_2} + U_{\gamma_2} \subseteq U_{\gamma_1}$ and $U_{\gamma_3} \cdot U_{\gamma_3} \subseteq U_{\gamma_2}$.

Since $\tau_1 < \tau_2$, then there exists an element $\beta_3 \in \mathrm{B}$ such that $V_{\beta_3} \subseteq U_{\gamma_3} \bigcap V_{\beta_2}$. If $a \in V_{\beta_3} + (U_{\gamma_3} \bigcap I)$ and $b \in V_{\beta_3} + (U_{\gamma_3} \bigcap I)$, then there are elements a_1 and b_1 from the set V_{β_3} and elements a_2 and b_2 from $U_{\gamma_3} \bigcap I$ such that $a = a_1 + b_1$ and $b = a_2 + b_2$. Then $a \cdot b = (a_1 + b_1) \cdot (a_2 + b_2) = a_1 \cdot a_2 + a_1 \cdot b_2 + b_1 \cdot a_2 + b_1 \cdot b_2$, where $a_1 \cdot a_2 \in V_{\beta_3} \cdot V_{\beta_3} \subseteq V_{\beta_1}$ and

$$a_1 \cdot b_2 + b_1 \cdot a_2 + b_1 \cdot b_2 \in V_{\beta_3} \cdot (U_{\gamma_3} \bigcap I) + (U_{\gamma_3} \bigcap I) \cdot V_{\beta_3} + (U_{\gamma_3} \bigcap I) \cdot (U_{\gamma_3} \bigcap I) \subseteq$$

$$(U_{\gamma_3} \cdot U_{\gamma_3} + U_{\gamma_3} \cdot U_{\gamma_3} + U_{\gamma_3} \cdot U_{\gamma_3}) \bigcap I \subseteq (U_{\gamma_2} + U_{\gamma_2} + U_{\gamma_2}) \bigcap I \subseteq U_{\gamma_1} \bigcap I, \text{ and hence}$$

$$a \cdot b = (a_1 + b_1) \cdot (a_2 + b_2) = a_1 \cdot a_2 + a_1 \cdot b_2 + b_1 \cdot a_2 + b_1 \cdot b_2 \in V_{\beta_1} + (U_{\gamma_1} \bigcap I).$$

Since the elements a and b are arbitrary, then it follows that

$$(V_{\beta_3} + (U_{\gamma_3} \bigcap I)) \cdot (V_{\beta_3} + (U_{\gamma_3} \bigcap I)) \subseteq V_{\beta_1} + (U_{\gamma_1} \bigcap I).$$

Consequently, the family Ω also satisfies condition 5, which means that the family Ω is a basis of filter of neighborhoods of zero in the topological ring (Q, τ) for some ring topology τ.

Since $V_\beta \subseteq V_\beta + (U_\gamma \bigcap I)$ for any element $\beta \in \mathrm{B}$ and $\gamma \in \Gamma$, then $\tau \leq \tau_2$.

Moreover, if $\gamma \in \Gamma$, then there exists an element $\gamma_1 \in \Gamma$ such that $U_{\gamma_1} + U_{\gamma_1} \subseteq U_\gamma$, and since $\tau_1 < \tau_2$, there exists an element $\beta \in \mathrm{B}$ such that $V_\beta \subseteq U_{\gamma_1}$. Then $V_\beta + (U_{\gamma_1} \bigcap I) \subseteq U_{\gamma_1} + U_{\gamma_1} \subseteq U_\gamma$, and the arbitrariness of the element $\gamma \in \Gamma$ implies that $\tau_1 \leq \tau$.

So, we have proved that $\tau_1 \leq \tau \leq \tau_2$.

Since $\tau_1 \prec_\Delta \tau_2$, then $\tau_1 = \tau$ or $\tau = \tau_2$, and because $\sup\{\tau_1, \tau(I)\} \neq \sup\{\tau_2, \tau(I)\}$, then $\sup\{\tau_1, \tau(I)\} < \sup\{\tau_2, \tau(I)\}$, and so there exists a neighborhood V_{β_1} of zero in the topological ring $(R, \sup\{\tau_2, \tau(I)\})$ such that $U_\gamma \bigcap I \not\subseteq V_{\beta_1} \bigcap I$ for any $\gamma \in \Gamma$.

Since $((U_\gamma \bigcap I) + V_\beta) \bigcap I = (U_\gamma \bigcap I) + (V_\beta \bigcap I) \supseteq U_\gamma \bigcap I$, then $((U_\gamma \bigcap I) + V_\beta) \bigcap I \not\subset V_{\beta_1}$, for any $\gamma \in \Gamma$, and therefore $\tau_1 \leq \tau < \tau_2$.

Since $\tau_1 \prec \tau_2$, then $\tau = \tau_1$, and hence $\{(U_\gamma \bigcap I) + V_\beta | \gamma \in \Gamma, \beta \in \mathrm{B}\}$ is a basis of the filter of neighborhoods of zero in the topological ring (R, τ_1) .

Note that if Δ is the lattice of all ring topologies on the ring Q in each of which the topological ring has a basis of the filter of neighborhoods of zero which consists of subgroups of additive group of ring and $\tau_1, \tau_2 \in \Delta$, then $\tau \in \Delta$.

This completes the proof of the theorem.

Corollary 3.3. *Let Δ be the lattice of all ring topologies on the ring Q or the lattice of all ring topologies in each of which the topological ring has a basis of the filter of neighborhoods of zero which consists of subgroups of the additive group of the ring Q and let τ_1 and τ_2 be ring topologies on R such that $\tau_1 \prec_\Delta \tau_2$. If* I *is an ideal of the ring Q such that* $sup\{\tau_1, \tau(\mathrm{I})\} \neq \sup\{\tau_2, \tau(\mathrm{I})\}$[1] *and* I *is an open ideal*

[1] notation τ(I) see in Remark 2.10

in the topological ring (Q, τ_2)*, then* I *is an open ideal in the topological ring* (Q, τ_1) too.

Indeed, since the ideal I is an open ideal in the topological ring $(Q\tau_2)$, then the ideal I is a neighborhood of zero in topological ring (Q, τ_2). Then, according to Theorem 3.2, the set $I = I + (Q \cap I)$ will be a neighborhood of zero in the topological ring (Q, τ_1), and hence the ideal I will be open ideal in the topological ring (Q, τ_1).

Remark 3.4. Will build a ring R, an ideal I of the ring R and two ring topologies τ_1 and τ_2 in each of which the topological ring has a basis of filter of neighborhoods of zero which consists of subgroups of the additive group of the ring such that $\tau_1 \prec \tau_2$ and between topologies $sup\{\tau(I), \tau_1\}$ and $sup\{\tau(I), \tau_2\}$ there are an infinite number of ring topologies in each of which the topological ring has a basis of filter of neighborhoods of zero which consists of subgroups of the additive group of the ring.

Example 3.5.[2]

Remark 3.5.1. It is easy to see that the following statements are true:

– since $R_j \cdot R_k = (0,0)$ if $j \neq k$, then $\widetilde{r} \cdot h_i = pr_i(\widetilde{r}) \cdot h_i$ for any element $\widetilde{r} \in \widetilde{R} = \sum_{j=1}^{\infty} R_j$, any natural number i, and any element $h_i \in R_i$;

– for any set $A \in \mathfrak{N}$ the set $\widetilde{W_A}$ is an ideal of the ring $\widetilde{R}$;

– for any set $A \in \mathfrak{N}$ the set $\widetilde{W'_A}$ is a subring of the ring $\widetilde{R}$;

– for any subring M of the ring $\mathbb{R}$ and any set $A \in \mathfrak{N}$ the set $\widetilde{W_{A,M}}$ is a subring of the ring $\widetilde{R}$.

Proposition 3.5.2. If $\mathfrak{M}$ is the lattice of all ring topologies of the ring $\widetilde{R}$ in each of which the topological ring has a basis of filter of neighborhoods of zero consisting of subgroups of the additive group of the ring $\widetilde{R}$, then the following statements are true:

3.5.2.1. The family $\{W_A | A \in \mathfrak{N}\}$ is a basis of filter of neighborhoods of zero for some ring topology $\widetilde{\tau}_1 \in \mathfrak{M}$ of the ring $\widetilde{R}$;

3.5.2.2. The family $\{W'_A | A \in \mathfrak{N}\}$ is a basis of filter of neighborhoods of zero for some ring topology $\widetilde{\tau}_0 \in \mathfrak{M}$ of the ring $\widetilde{R}$;

3.5.2.3. For any subring M of the ring R the family $\{V_{A,M} | A \in \mathfrak{N}\}$ is a basis of filter of neighborhoods of zero for some ring topology $\widetilde{\tau}_M \in \mathfrak{M}$ of the ring $\widetilde{R}$;

3.5.2.4. $\widetilde{\tau}_1 \prec \widetilde{\tau}_0$, i.e. the topology $\widetilde{\tau}_0$ is a covering of the topology $\widetilde{\tau}_1$ in the lattice $\mathfrak{M}$.

Proof of Statement 3.5.2.1. Since $\mathfrak{N}$ is a filter and since for each $A \in \mathfrak{N}$ the set W_A is an ideal of the ring $\widetilde{R}$, then the set $\{W_A | A \in \mathfrak{N}\}$ satisfies conditions 1 – 6 of Theorem 2.8 and hence it is a basis of filter of neighborhoods of zero for some ring topology $\widetilde{\tau}_1$ of the ring $\widetilde{R}$, and hence Statement 2.5.2.1 is proved.

[2]See the notations 2.1 – 7.7 used for presenting further results

Proof of Statement 3.5.2.2. Since $\widetilde{W}_A$ is a subring of the ring $\widetilde{R}$ for any subset $A \in \mathfrak{N}$ and $\widetilde{W}_B \subseteq \widetilde{W}_A$ for any A and $B \in \mathfrak{N}$ such that $B \subseteq A$, then for the family $\{W'_A | A \in \mathfrak{N}\}$ properties $1-5$ of Theorem 2.8 hold.

The fact that $\mathfrak{N}$ is a free filter implies that for any element $\widetilde{r} \in \widetilde{R}$ and any set $A \in \mathfrak{N}$ there exists a set $B \in \mathfrak{N}$ such that $B \subseteq A$ and $pr_i(\widetilde{r}) = (0,0)$ for any natural number $i \in B$. Then $\widetilde{g} \cdot \widetilde{r} = \widetilde{g} \cdot \widetilde{r} = \widetilde{0} \in \widetilde{W}'_A$ for any element $\widetilde{g} \in \widetilde{W}'_B$, and hence property 6 of Theorem 2.8 holds. Therefore, the family $\{W'_A | A \in \mathfrak{N}\}$ is a basis of filter of neighborhoods of zero for some ring topology $\widetilde{\tau}_0$, and so Statement 3.5.2.2 is proved.

The proof of Statemen 3.5.2.3 is carried out similarly to the proof of Statement 3.5.2.2.

Proof of Statement 3.5.2.4. Since $\widetilde{W}'_A \subseteq \widetilde{W}_A$ for any set $A \in \mathfrak{N}$, then $\widetilde{\tau}_0 \leq \widetilde{\tau}_1$. It is easy to see that $\widetilde{\tau}_0 \neq \widetilde{\tau}_1$.

Let now $\widetilde{\tau}$ be a ring topology on the ring $\widetilde{R}$ such that $\widetilde{\tau}_0 \leq \tau \leq \widetilde{\tau}_1$ and let $\widetilde{\tau} \neq \widetilde{\tau}_1$.

Show that for any set $C \in \mathfrak{N}$ and any neighborhood $\widetilde{U}$ of zero of the topological ring $(\widetilde{R}, \widetilde{\tau})$ there exists an element $\widetilde{g} \in \widetilde{U}$ such that $pr_i(\widetilde{g}) \notin R'_i$ for some natural number $i \in C$.

Let $C \in \mathfrak{N}$ and let $\widetilde{U}$ be a neighborhood of zero of the topological ring $(\widetilde{R}, \widetilde{\tau})$. Since $\widetilde{\tau} \leq \widetilde{\tau}_1$ and $\widetilde{\tau} \neq \widetilde{\tau}_1$, then there exists a set $A \in \mathfrak{N}$ such that $A \subseteq C$ and $\widetilde{W}'_A \subseteq \widetilde{U}$, and such that $\widetilde{W}'_A$ is not a neighborhood of zero in the topological ring $(\widetilde{R}, \widetilde{\tau})$. Since $\widetilde{\tau}_0 \leq \widetilde{\tau}$, the set $\widetilde{W}_A$ is a neighborhood of zero in the topological ring $(\widetilde{R}, \widetilde{\tau})$. Then $\widetilde{W}_A \bigcap \widetilde{U}$ will be a neighborhood of zero in the topological ring $(\widetilde{R}, \widetilde{\tau})$, and since $\widetilde{W}'_A$ is not a neighborhood of zero in the topological ring $(\widetilde{R}, \widetilde{\tau})$, then $\widetilde{W}_A \bigcap \widetilde{U} \nsubseteq \widetilde{W}'_A$.

Since $\widetilde{W}_A \bigcap (\sum_{i=1}^{\infty} R'_i) = \widetilde{W}'_A$, then $\widetilde{W}_A \bigcap \widetilde{U}$ is not contained in the set $\sum_{i=1}^{\infty} R'_i$, and hence there exists an element $\widetilde{g} \in \widetilde{W}_A \bigcap \widetilde{U}$ such that $\widetilde{g} \notin \sum_{i=1}^{\infty} R'_i$ for some natural number $i \in A \subseteq C$.

Thus, we have proved that for any set $C \in \mathfrak{N}$ and any neighborhood $\widetilde{U}$ of zero of the topological ring $(\widetilde{R}, \widetilde{\tau})$ there exists an element $\widetilde{g} \in \widetilde{U}$ such that $pr_i(\widetilde{g}_0) \notin R'_i$ for some natural number $i \in C$.

Let us show that $B_U = \{i \in \mathbb{N} |$ there exists $\widetilde{g}_i \in \widetilde{U}$ such that $pr_i(\widetilde{g}_i) \notin R'_i\} \in \mathfrak{N}$. Because $B_U \bigcap C \neq \emptyset$, from the arbitrariness of the set $C \in \mathfrak{N}$ and the fact that the set $\mathfrak{N}$ is an ultrafilter it follows that $B_U \in \mathfrak{N}$.

Thus, we have shown that $B_U \in \mathfrak{N}$ for any neighborhood $\widetilde{U}$ of zero of the topological ring $(\widetilde{R}, \widetilde{\tau})$.

Let us show that $\widetilde{\tau}_0 = \widetilde{\tau}$. Since $\widetilde{\tau}_0 \leq \widetilde{\tau}$, it suffices to prove that any neighborhood $\widetilde{U}_0$ of zero of the topological ring $(\widetilde{R}, \widetilde{\tau})$ is a neighborhood of zero of the topological ring $(\widetilde{R}, \widetilde{\tau}_0)$. There exists a neighborhood $\widetilde{U}_1$ of zero of the topological ring $(\widetilde{R}, \widetilde{\tau})$ such that $\widetilde{U}_1 \cdot \widetilde{U}_1 \subseteq \widetilde{U}_0$. Without loss of generality, we may assume that sets $\widetilde{U}_0$ and $\widetilde{U}_1$ are subgroups of the additive group of the ring $\widetilde{R}$.

Since $\widetilde{\tau} < \widetilde{\tau}_1$, then there exists a set $D \in \mathfrak{N}$ such that $\widetilde{W}'_D \subseteq \widetilde{U}_1$, and then $D \subseteq B_{U_1}$.

Let us show that $\widetilde{W}_D \subseteq \widetilde{U}_0$. Let $\widetilde{g}_0 \in \widetilde{W}_D$. Then there exists a finite set $\{i_1, \dots i_n\} \subseteq D$ of natural numbers such that $\widetilde{g}_0 \in \sum_{j=1}^{n} R_{i_j}$, and for every natural number $1 \leq j \leq n$ there is such an element $r_{i_j} = (a_{i_j}, b_{i_j}) \in R_{i_j} \subseteq \widetilde{R}$ that $\widetilde{g}_0 = \sum_{j=1}^{n} r_{i_j}$.

For any natural number $i \in D$ there exists an element $\widetilde{q}_i \in \widetilde{U}_1$ such that $pr_i(\widetilde{q}_i) \notin R'_i$, and hence $pr_i(\widetilde{q}_i) = (q_i, q'_i)$, where $q_i \neq 0$.

For each natural number $1 \leq j \leq n$ consider an element $\widetilde{h}_{i_j} \in \widetilde{R}$ such that $pr_k(\widetilde{h}_{i_j}) = (0,0)$ if $k \neq i_j$ and $pr_k(\widetilde{h}_{i_j}) = (0, a_{i_j} \cdot q_{i_j}^{-1})$. Since $i_j \in D$ then $\widetilde{h}_{i_j} \in \widetilde{R}'_{i_j} \subseteq \sum_{i \in D} R'_i = \widetilde{W}'_D \subseteq \widetilde{U}_1$. Then

$$r_{i_j} = (a_{i_j}, b_{i_j}) = (q_{i_j} \cdot a_{i_j} \cdot q_{i_j}^{-1}, q'_{i_j} \cdot a_{i_j} \cdot q_{i_j}^{-1}) + (0, b_{i_j} - q'_{i_j} \cdot a_{i_j} \cdot q_{i_j}^{-1}) =$$

$$(q_{i_j} \cdot 0 + q_{i_j} \cdot a_{i_j} \cdot q_{i_j}^{-1} + q'_{i_j} \cdot 0, q'_{i_j} \cdot a_{i_j} \cdot q_{i_j}^{-1}) + (0, b_{i_j} - q'_{i_j} \cdot a_{i_j} \cdot q_{i_j}^{-1}) =$$

$(q_{i_j}, q'_{i_j}) \cdot (0, a_{i_j} \cdot q_{i_j}^{-1}) = \widetilde{q}_{i_j} \cdot (0, a_{i_j} \cdot q_{i_j}^{-1}) \in \widetilde{U}_1 \cdot R'_{i_j} \subseteq \widetilde{U}_1 \cdot (\sum_{i \in D} R'_i) \subseteq U_1 \cdot U_1 \subseteq \widetilde{U}_0$ for any natural number $1 \leq j \leq n$.

Since U_0 is a subgroup of the additive group of the ring $\widetilde{R}$, then $\widetilde{g}_0 = \sum_{j=1}^{n} r_{i_j} \in U_0$.

Since the element $\widetilde{g}_0$ is arbitrary, it follows that $\widetilde{W}_D \subseteq \widetilde{U}_0$, and hence $\widetilde{U}_0$ is a neighborhood of zero of the topological ring $(\widetilde{R}, \widetilde{R}\tau_0)$, and since $\widetilde{U}_0$ is arbitrary, it follows that $\widetilde{\tau}_0 = \widetilde{\tau}$.

So we have proved that $\widetilde{\tau}_1 \prec \widetilde{\tau}_0$. This proves Statement 3.5.2.4.

Remark 3.5.3. It is easy to see that $\widetilde{I} = \{\widetilde{r} \in \widetilde{R} | pr_i(\widetilde{r}) = (q, 0),$ where $q \in \mathbb{Q}$ and $i \in \mathbb{N}\}$ is an ideal of the ring $\widetilde{R}$ and

$$sup\{\widetilde{\tau}_1, \tau(\widetilde{I})\} < \widetilde{\tau}_{M_1} < \widetilde{\tau}_{M_2} < \ldots < \tau(\{0\}) = sup\{\widetilde{\tau}_0, \tau(\widetilde{I})\}$$

for any decreasing sequence $M_1 \supset M_2 \supset \ldots$ of subrings of the ring $\mathbb{R}$.

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