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The q.Zariski topology on the quasi-primary spectrum of a ring

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Abstract. Let R be a commutative ring with identity. We topologize q.Spec(R), the quasi-primary spectrum of R, in a way similar to that of defining the Zariski topology on the prime spectrum of R, and investigate the properties of this topological space. Rings whose q.Zariski topology is respectively T_0, T_1, irreducible or Noetherian are studied, and several characterizations of such rings are given.

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1 Introduction

Let R denote a commutative ring with identity. The Zariski topology on the prime spectrum Spec(R), the set of prime ideals of R, play an important role in the fields of commutative algebra, algebraic geometry and lattice theory. For each ideal I of R, the set $V(I) = \{p \in \mathrm{Spec}(R) \mid p \supseteq I\}$ satisfies the axioms for the closed sets of the Zariski topology on Spec(R) (see for example, Atiyah and Macdonald [1]). In the literature, there are many different topologies of commutative or noncommutative rings ([2, 5, 6]).

About a quarter of a century later, in [3] the notion of quasi-primary ideals as a generalization of the notion of primary ideals was introduced. A proper ideal q of R is called quasi-primary if $rs \in q$, for $r, s \in R$, implies that either $r \in \sqrt{q}$ or $s \in \sqrt{q}$. Equivalently, q is quasi-primary if and only if $\sqrt{q}$ is prime [3, Definition 2, p. 176]. In this case, q is said to be p-quasi-primary where $p = \sqrt{q}$. In the sequel, we introduce and study a topology on quasi-primary spectrum q.Spec(R), the set of all quasi-primary ideals of R, such that the Zariski topology is a subspace of this topology. We investigate the interplay between the properties of this space and the algebraic properties of the ring under consideration. In particular, assuming suitable conditions for each result, we investigated when this space is T_0 (Theorem 4(4)), T_1 (Theorem 4(5)), Noetherian (Theorem 5) or irreducible (Theorem 6 and Corollary 1).

2 Main Results

Throughout, R is a commutative ring with $1_R \neq 0_R$. We denote the set of all quasi-primary ideals of R by q.Spec(R) and define the variety of an ideal I of R as follows:

$$V^{\mathbf{q}}(I) = \{q \in \text{q.Spec}(R) \mid \sqrt{q} \supseteq I\}.$$

The following lemma shows that the set $\mathcal{T}(R) = \{V^{\mathbf{q}}(I) \mid I \text{ is an ideal of } R\}$ satisfies the axioms for closed sets in a topological space on q.Spec(R), called q.Zariski topology.

The proof of the next result is easy and so it is omitted.

Lemma 1. *For any ideals I, J and I_λ $(\lambda \in \Lambda)$ of a ring R, the following hold.*

(1) $V^{\mathbf{q}}(R) = \emptyset$ *and* $V^{\mathbf{q}}(0) = \text{q.Spec}(R)$.

(2) $\bigcap\limits_{\lambda\in\Lambda} V^{\mathbf{q}}(I_\lambda) = V^{\mathbf{q}}(\sum\limits_{\lambda\in\Lambda} I_\lambda)$.

(3) $V^{\mathbf{q}}(I) \cup V^{\mathbf{q}}(J) = V^{\mathbf{q}}(I \cap J)$.

Let Y be a subset of q.Spec(R) for a ring R. We will denote the intersection of all elements in Y by $\xi(Y)$ and the closure of Y in q.Spec(R) with respect to the q.Zariski topology by $cl(Y)$. Also the set of all p-quasi-primary ideals of a ring R is denoted by q.Spec$_p(R)$.

Next we offer some descriptions for the two proper ideals I and J of R that will be useful in the sequel.

Lemma 2. *Let I and J be proper ideals of a ring R. Then the following hold.*

(1) $V^{\mathbf{q}}(I) = V^{\mathbf{q}}(\sqrt{I})$.

(2) $V^{\mathbf{q}}(I) \subseteq V^{\mathbf{q}}(J)$ *if and only if* $\sqrt{J} \subseteq \sqrt{I}$, *and if* $J \subseteq I$, *then* $V^{\mathbf{q}}(I) \subseteq V^{\mathbf{q}}(J)$.

(3) $V^{\mathbf{q}}(I) = \bigcup\limits_{I\subseteq p\in Spec(R)} \text{q.Spec}_p(R)$.

(4) Let Y be a subset of q.Spec(R). *Then* $Y \subseteq V^{\mathbf{q}}(I)$ *if and only if* $I \subseteq \sqrt{\xi(Y)}$.

Consider the surjective map $\phi : \text{q.Spec}(R) \to \text{Spec}(R)$ given by $\phi(q) = \sqrt{q}$ for every $q \in \text{q.Spec}(R)$. In the following result we ghather some properties of this map.

Proposition 1. *Let R be a ring.*

(1) The map ϕ is continuous with respect to the q.Zariski topology; more precisely, $\phi^{-1}(V(I)) = V^{\mathbf{q}}(I)$ for every ideal I of R.

(2) $\phi(V^{\mathbf{q}}(I)) = V(I)$ *and* $\phi(\text{q.Spec}(R) - V^{\mathbf{q}}(I)) = \text{Spec}(R) - V(I)$ *i.e. ϕ is both closed and open.*

(3) ϕ is injective if and only if it is a homeomorphism.

Proof. (1). Let I be an ideal of R. Then

$$\begin{aligned} q \in \phi^{-1}(V(I)) &\Leftrightarrow \phi(q) \in V(I) \\ &\Leftrightarrow \sqrt{q} \supseteq I \\ &\Leftrightarrow q \in V^{\mathbf{q}}(I). \end{aligned}$$

(2). As we have seen in (1), $\phi(V^{\mathbf{q}}(I)) = \phi(\phi^{-1}(V(I))) = \phi \circ \phi^{-1}(V(I)) = V(I)$ as ϕ is surjective. Similarly,

$$\begin{aligned} \phi(\mathrm{q.Spec}(R) - V^{\mathbf{q}}(I)) &= \phi(\phi^{-1}(\mathrm{Spec}(R)) - \phi^{-1}(V(I))) \\ &= \phi(\phi^{-1}(\mathrm{Spec}(R) - V(I))) \\ &= \phi \circ \phi^{-1}(\mathrm{Spec}(R) - V(I)) \\ &= \mathrm{Spec}(R) - V(I). \end{aligned}$$

(3). This follows from (2). □

Theorem 1. *For any ring R, the following are equivalent:*

(1) q.Spec(R) *is connected;*

(2) Spec(R) *is connected;*

(3) The ring R contains no idempotent other than 0 *and* 1.

Proof. (1) $\Rightarrow$ (2). Suppose q.Spec(R) is a connected space. By Proposition 1, the map ϕ is surjective and continuous and so Spec(R) is also a connected space.

(2) $\Rightarrow$ (1). Suppose, on the contrary, that q.Spec(R) is disconnected. There exists a non-empty proper subset W of q.Spec(R) that is both closed and open. By Proposition 1, $\phi(W)$ is a non-empty subset of Spec(R) that is also clopen. To complete the proof, it suffices to show that $\phi(W)$ is a proper subset of Spec(R), and so Spec(R) is disconnected, a contradiction. Since W is an open set, we have $W = \mathrm{q.Spec}(R) - V^{\mathbf{q}}(I)$ for some ideal I of R and hence Proposition 1 shows that $\phi(W) = \mathrm{Spec}(R) - V(I)$. It follows that $\phi(W)$ is a proper subset of Spec(R). Otherwise, if $\phi(W) = \mathrm{Spec}(R)$, then $V(I) = \emptyset$, and so $I = R$. We conclude from this fact that $W = \mathrm{q.Spec}(R)$ which is impossible.
(2) $\Leftrightarrow$ (3) is a well-known fact, for example [1, Exercise 22, p.14]. □

For any ideal I of R, we define $\Lambda_R(I) = \mathrm{q.Spec}(R) - V^{\mathbf{q}}(I)$ as an open set of q.Spec(R). Also for any $a \in R$, we mean $\Lambda_R(a)$ by $\Lambda_R(Ra)$. Clearly, $\Lambda_R(0) = \emptyset$ and $\Lambda_R(1) = \mathrm{q.Spec}(R)$. Following result shows that the set $B = \{\Lambda_R(a) \mid a \in R\}$ is a base for the q.Zariski topology on q.Spec(R).

Theorem 2. *Let R be a ring and $B = \{\Lambda_R(a) \mid a \in R\}$. Then the set B forms a base for the q.Zariski topology on* q.Spec(R).

Proof. We may assume that $\text{q.Spec}(R) \neq \emptyset$. Let O be an open subset in $\text{q.Spec}(R)$. Thus $O = \text{q.Spec}(R) - V^{\mathbf{q}}(I)$ for some ideal I of R. Therefore

$$\begin{aligned} O &= \text{q.Spec}(R) - V^{\mathbf{q}}(I) = \text{q.Spec}(R) - V^{\mathbf{q}}(\sum_{a\in I} Ra) \\ &= \text{q.Spec}(R) - \bigcap_{a\in I} V^{\mathbf{q}}(Ra) \\ &= \bigcup_{a\in I} \Lambda_R(a). \end{aligned}$$

It follows that the set B forms a base for the q.Zariski topology on $\text{q.Spec}(R)$. □

Theorem 3. *Let R be a ring and $a, b \in R$.*

(1) $\Lambda_R(a) = \emptyset$ if and only if a is a nilpotent element of R.

(2) $\Lambda_R(a) = \text{q.Spec}(R)$ if and only if a is a unit element of R.

(3) For each pair of ideals I and J of R, $\Lambda_R(I) = \Lambda_R(J)$ if and only if $\sqrt{I} = \sqrt{J}$ if and only if $V^{\mathbf{q}}(I) = V^{\mathbf{q}}(J)$.

(4) $\Lambda_R(ab) = \Lambda_R(a) \cap \Lambda_R(b)$.

(5) $\text{q.Spec}(R)$ is quasi-compact.

(6) For any $c \in R$, $\Lambda_R(c)$ is qusi-compact, that is, every open covering of $\Lambda_R(c)$ has a finite subcovering.

(7) An open subset of $\text{q.Spec}(R)$ is quasi-compact if and only if it is a finite union of sets $\Lambda_R(a)$.

Proof. (1). Let $a \in R$. Then

$$\begin{aligned} \emptyset &= \Lambda_R(a) = \text{q.Spec}(R) - V^{\mathbf{q}}(Ra) \\ &\Leftrightarrow V^{\mathbf{q}}(Ra) = \text{q.Spec}(R) \\ &\Leftrightarrow \sqrt{q} \supseteq Ra \ for\ every\ q \in \text{q.Spec}(R) \\ &\Leftrightarrow a\ is\ in\ every\ prime\ ideal\ of\ R \\ &\Leftrightarrow a\ is\ a\ nilpotent\ element\ of\ R. \end{aligned}$$

(2).

$$\begin{aligned} \Lambda_R(a) &= \text{q.Spec}(R) \\ &\Leftrightarrow a \notin \sqrt{q}\ for\ all\ q \in \text{q.Spec}(R) \\ &\Rightarrow a \notin q\ for\ all\ q \in Max(R) \\ &\Rightarrow a\ is\ unit. \end{aligned}$$

Conversely, it is clear that a unit element a of R is not contained in any quasi-primary ideal of R. That is, $\Lambda_R(a) = \text{q.Spec}(R)$.

(3) is clear by Lemma 2(2).
(4). Let $q \in V^{\mathbf{q}}(Rab)$. Then

$$\begin{aligned}\sqrt{q} \supseteq \sqrt{Rab} &= \sqrt{Ra} \cap \sqrt{Rb} \\ &\Leftrightarrow \sqrt{q} \supseteq \sqrt{Ra} \textit{ or } \sqrt{q} \supseteq \sqrt{Rb} \\ &\Leftrightarrow q \in V^{\mathbf{q}}(Ra) \textit{ or } q \in V^{\mathbf{q}}(Rb) \\ &\Leftrightarrow q \in V^{\mathbf{q}}(Ra) \cup V^{\mathbf{q}}(Rb).\end{aligned}$$

It follows that $V^{\mathbf{q}}(Rab) = V^{\mathbf{q}}(Ra) \cup V^{\mathbf{q}}(Rb)$, as required.
(5). Let $\text{q.Spec}(R) = \underset{i \in I}{\cup} \Lambda_R(J_i)$, where $\{J_i\}_{i \in I}$ is a family of ideals of R. We clearly have $\Lambda_R(R) = \text{q.Spec}(R) = \Lambda_R(\sum_{i \in I} J_i)$. Thus, by the part (3), $R = \sqrt{\sum_{i \in I} J_i}$ and hence, $1 \in \sum_{i \in I} J_i$. So there exist $i_1, i_2, \cdots, i_n \in I$ such that $1 \in \sum_{k=1}^{n} J_{i_k}$, that is $R = \sum_{k=1}^{n} J_{i_k}$. Consequently, $\text{q.Spec}(R) = \Lambda_R(R) = \Lambda_R(\sum_{k=1}^{n} J_{i_k}) = \overset{n}{\underset{k=1}{\cup}} \Lambda_R(J_{i_k})$.
(6). Let $c \in R$. For any open covering of $\Lambda_R(c)$, there is a family $\{a_i \mid a_i \in R, i \in I\}$ of elements of R such that $\Lambda_R(c) \subseteq \underset{i \in I}{\cup} \Lambda_R(a_i)$, since $B = \{\Lambda_R(a_i) \mid a_i \in R, i \in I\}$ forms a base for the q.Zariski topology on $\text{q.Spec}(R)$, by Theorem 2.
It is clear that the map $\phi : \text{q.Spec}(R) \to \text{Spec}(R)$ given by $\phi(q) = \sqrt{q}$ is surjective, and so there exists a finite subset I' of I such that $\Lambda_R(c) \subseteq \underset{i \in I'}{\cup} \Lambda_R(a_i)$, because $\phi(\Lambda_R(a)) = \text{Spec}(R) - V(a)$ is quasi-compact by [1, Exercise 1.17 p. 12]
(7). The sufficiency follows by exactly the same argument as (6). For the necessity, if an open subspace Y of $\text{q.Spec}(R)$ is a union of a finite number of sets $\Lambda_R(Ra)$, then any open cover $\{\Lambda_R(Ra_i)\}_{i \in I}$ of Y induces an open cover for each of the $\Lambda_R(Ra)$. By (6), each of those will have a finite subcover and these subcovers yield a finite subcover of $\text{q.Spec}(R)$.

□

A topological space $(X; \tau)$ is said to be a T_0-space if for each pair of distinct points a, b in X, either there exists an open set containing a and not b, or there exists an open set containing b and not a. It has been shown that a topological space is T_0 if and only if the closures of distinct points are distinct. Also, a topological space $(X; \tau)$ is called a T_1-space if every singleton set $\{x\}$ is closed in $(X; \tau)$. Clearly every T_1-space is a T_0-space.

Theorem 4. *Let R be a ring, $Y \subseteq \text{q.Spec}(R)$ and let $q \in \text{q.Spec}_p(R)$. Then*

(1) $V^{\mathbf{q}}(\xi(Y)) = cl(Y)$. In particular, $cl(\{q\}) = V^{\mathbf{q}}(q)$.

(2) If $(0) \in Y$, then Y is dense in $\text{q.Spec}(R)$.

(3) The set $\{q\}$ is closed in $\text{q.Spec}(R)$ if and only if

(i) p is a maximal element in $\{\sqrt{q'} \mid q' \in \text{q.Spec}(R)\}$, and

(ii) $\text{q.Spec}_p(R) = \{q\}$.

(4) The following statements are equivalent:

(i) $\text{q.Spec}(R)$ *is a* T_0*-space;*

(ii) the map $\phi : \text{q.Spec}(R) \to \text{Spec}(R)$*, given by* $\phi(q) = \sqrt{q}$*, is injective;*

(iii) $\text{q.Spec}(R) = \text{Spec}(R)$.

(5) $\text{q.Spec}(R)$ *is a* T_1*-space if and only if* $\text{q.Spec}(R)$ *is a* T_0*-space and* $\text{q.Spec}(R) = \text{Spec}(R) = Max(R)$ *(where* $Max(R)$ *is the set of all maximal ideals of* R*).*

(6) Let $(0) \in \text{q.Spec}(R)$*. Then* $\text{q.Spec}(R)$ *is a* T_1*-space if and only if* (0) *is the only quasi-primary ideal of* R*.*

(7) Let R *be a domain. If* $\text{q.Spec}(R)$ *is a* T_1*-space, then* R *is a field.*

Proof. (1). Let $q \in Y$. Then $\xi(Y) \subseteq q \subseteq \sqrt{q}$. Therefore $q \in V^{\mathbf{q}}(\xi(Y))$ and so $Y \subseteq V^{\mathbf{q}}(\xi(Y))$. Next, let $V^{\mathbf{q}}(I)$ be any closed subset of $\text{q.Spec}(R)$ containing Y. Then $\sqrt{q} \supseteq I$ for every $q \in Y$ and hence $\sqrt{\xi(Y)} \supseteq I$.
It follows that $\sqrt{q'} \supseteq \sqrt{\xi(Y)} \supseteq I$ for every $q' \in V^{\mathbf{q}}(\xi(Y))$ and so $V^{\mathbf{q}}(\xi(Y)) \subseteq V^{\mathbf{q}}(I)$. Thus $V^{\mathbf{q}}(\xi(Y))$ is the smallest closed subset of $\text{q.Spec}(R)$ containing Y , hence $V^{\mathbf{q}}(\xi(Y)) = cl(Y)$.
(2) is trivial by (1).
(3). Suppose that $\{q\}$ is closed. Then, by (1), $\{q\} = V^{\mathbf{q}}(q)$. Assue that $q' \in \text{q.Spec}(R)$ such that $\sqrt{q'} \supseteq p$. Hence, $q' \in V^{\mathbf{q}}(q) = \{q\}$, and so $\text{q.Spec}_p(R) = \{q\}$. Conversely, assume that (i) and (ii) hold. Let $q' \in cl(\{q\})$. Then $\sqrt{q'} \supseteq q$ by (1). It follows from (i) that $\sqrt{q'} = \sqrt{q} = p$ and hence $q' = q$ by (ii). This yields $cl(\{q\}) = \{q\}$.
(4). (i)⇒(ii) Suppose $q, q' \in \text{q.Spec}(R)$ such that $\sqrt{q} = \sqrt{q'}$ and $q \neq q'$. Since $\text{q.Spec}(R)$ is a T_0-space, there is an element $a \in R$ such that $q \in \Lambda_R(a)$ and $q' \notin \Lambda_R(a)$. Thus $\sqrt{q} \not\supseteq Ra$ and $\sqrt{q'} \not\supseteq Ra$, a contradiction. Thus the map ϕ is injective.
(ii)⇒(iii) is clearly true and (iii)⇒(i) will be obtained by [1, Exercise 18(iv) p. 13].
(5) is easy to check from the definition and the parts $(3), (4)$.
(6). Let $\text{q.Spec}(R)$ be a T_1-space. By the part (5), the ideal (0) is maximal and hence (0) is the only quasi-primary ideal of R. The converse follows from the definition and the part (3).
(7) follows from the part (6). □

A topological space X is said to be Noetherian if the open subsets of X satisfy the ascending chain condition. Since closed subsets are complements of open subsets, it comes to the same thing to say that the closed subsets of X satisfy the descending chain condition. Also a nonempty subset C of a topological space X is said to be irreducible if C can not be written as the union of two distinct closed sets.

Theorem 5. *Let R be a ring.*

(1) *If R is a Noetherian ring, then* q.Spec(R) *is a Noetherian topological space.*

(2) *$V^{\mathfrak{q}}(q)$ is an irreducible closed subset of* q.Spec(R) *for every quasi-primary ideal q of R.*

(3) *If I is an ideal of R such that $V^{\mathfrak{q}}(I)$ is an irreducible closed set, then there exists an irreducible ideal J of R such that $V^{\mathfrak{q}}(I) = V^{\mathfrak{q}}(J)$.*

(4) *If I is an ideal of R and* q.Spec(R) *is a Noetherian topological space, then $V^{\mathfrak{q}}(I) = \overset{k}{\underset{t=1}{\cup}} V^{\mathfrak{q}}(I_t)$ where $V^{\mathfrak{q}}(I_t)$ are irreducible closed sets and I_k are irreducible ideals of R.*

(5) *If I is an ideal of a Noetherian ring R, then $V^{\mathfrak{q}}(I)$ can be written as a finite union of irreducible closed sets $V^{\mathfrak{q}}(I_t)$, $1 \le t \le k$ such that for each t, I_t is an irreducible ideal of R.*

Proof. (1). Let $V^{\mathfrak{q}}(I_1) \supseteq V^{\mathfrak{q}}(I_2) \supseteq V^{\mathfrak{q}}(I_3) \supseteq \cdots$ be a chain of closed sets of q.Spec(R), where $\{I_t\}_{t=1}^{\infty}$ is a family of ideals of R. We conclude from Lemma 2(2) that $\sqrt{I_1} \subseteq \sqrt{I_2} \subseteq \sqrt{I_3} \subseteq \cdots$, and since R is a Noetherian ring, there exists a positive integer n such that for each positive integer $m \ge n$, $\sqrt{I_n} = \sqrt{I_m}$. Consequently, again by using Lemma 2(1), we have $V^{\mathfrak{q}}(I_n) = V^{\mathfrak{q}}(\sqrt{I_n}) = V^{\mathfrak{q}}(\sqrt{I_m}) = V^{\mathfrak{q}}(I_m)$, which completes the proof.
(2). It is clear that a singleton subset and its closure in q.Spec(R) are both irreducible. Now, the proof will be obtained by Theorem 4.
(3). Let $A = \{L \mid L$ is an ideal of R such that $V^{\mathfrak{q}}(I) = V^{\mathfrak{q}}(L)\}$. By Zorn's lemma, the set A has a maximal element, say J. We claim that J is irreducible. Assume, on the contrary, that $J = J_1 \cap J_2$ for some ideals J_1 and J_2 of R. Then $V^{\mathfrak{q}}(I) = V^{\mathfrak{q}}(J) = V^{\mathfrak{q}}(J_1 \cap J_2) = V^{\mathfrak{q}}(J_1) \cup V^{\mathfrak{q}}(J_2)$ and so $V^{\mathfrak{q}}(I)$ is equal to $V^{\mathfrak{q}}(J_1)$ or $V^{\mathfrak{q}}(J_2)$, since $V^{\mathfrak{q}}(I)$ is irreducible. It is a contradiction, since J is a maximal element of A and $J \subseteq J_1$ and $J \subseteq J_2$.
(4). According to [4, Exercise 4.11], every closed subset can be written as a union of finitely many irreducible closed sets in a Noetherian topological space. Now the part (3) completes the proof.
(5). By the part (1), q.Spec(R) is a Noetherian topological space and hence the assertion follows from the part (4). □

Theorem 6. *Let R be a ring and $Y \subseteq$* q.Spec(R)*. Then $\xi(Y)$ is a quasi-primary ideal of R if and only if Y is an irreducible space.*

Proof. Suppose $\xi(Y)$ is a quasi-primary ideal of R. Let $Y \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are two closed subsets of q.Spec(R). Then there exist two ideals I and J of R such that $Y_1 = V^{\mathfrak{q}}(I)$ and $Y_2 = V^{\mathfrak{q}}(J)$. Thus, $Y \subseteq V^{\mathfrak{q}}(I) \cup V^{\mathfrak{q}}(J) = V^{\mathfrak{q}}(I \cap J)$. It implies, by Lemma 2(4), that $I \cap J \subseteq \sqrt{\xi(Y)}$. It follows that either $I \subseteq \sqrt{\xi(Y)}$ or $J \subseteq \sqrt{\xi(Y)}$, since $\sqrt{\xi(Y)}$ is prime. Again by using Lemma 2(4), we conclude that either $Y \subseteq V^{\mathfrak{q}}(I) = Y_1$ or $Y \subseteq V^{\mathfrak{q}}(J) = Y_2$. Thus Y is irreducible. Conversely, assume that Y is an irreducible space. Let $ab \in \xi(Y)$ for some $a, b \in R$. Suppose, on the contrary, that $Ra \nsubseteq \sqrt{\xi(Y)}$ and $Rb \nsubseteq \sqrt{\xi(Y)}$. By Lemma 2(4), $Y \nsubseteq V^{\mathfrak{q}}(Ra)$ and $Y \nsubseteq V^{\mathfrak{q}}(Rb)$. Let $q \in Y$. Then $\sqrt{q} \supseteq \sqrt{\xi(Y)} \supseteq Rab$. This means that either $Ra \subseteq \sqrt{q}$ or $Rb \subseteq \sqrt{q}$. So, by Lemma 2(1),(2), we have either $V^{\mathfrak{q}}(q) \subseteq V^{\mathfrak{q}}(Ra)$ or $V^{\mathfrak{q}}(q) \subseteq V^{\mathfrak{q}}(Rb)$. Therefore, $Y \subseteq V^{\mathfrak{q}}(Ra) \cup V^{\mathfrak{q}}(Rb)$ and hence $Y \subseteq V^{\mathfrak{q}}(Ra)$ or $Y \subseteq V^{\mathfrak{q}}(Rb)$ as Y is irreducible. It is a contradiction. $\square$

Corollary 1. *Let R be a ring.*

(1) Let I be an ideal of R. Then $V(I)$ is irreducible in $\mathrm{q.Spec}(R)$ *if and only if $I \in$* $\mathrm{q.Spec}(R)$.

(2) If R is a domain, then $\mathrm{q.Spec}(R)$ *is irreducible.*

Proof. (1). Since $\sqrt{I} = \xi(V(I))$, Theorem 6 shows that $\sqrt{I}$ is quasi-primary if and only if $V(I)$ is irreducible. On the other hand, it is easy to see that $I \in \mathrm{q.Spec}(R)$ if and only if $\sqrt{I} \in \mathrm{q.Spec}(R)$. It completes the proof.
(2). Since (0) is a prime ideal of R, we have $\xi(\mathrm{q.Spec}(R)) \subseteq (\xi(\mathrm{Spec}(R)) = (0)$. Thus $\xi(\mathrm{q.Spec}(R))$ is a quasi-primary ideal of R and hence the result follows from Theorem 6. $\square$

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