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Postoptimal analysis of a finite cooperative game

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Abstract. We consider a finite cooperative game of several players with parameterized concept of equilibrium (optimality principles), when relations between players in coalition are based on the Pareto maximum. Introduction of this optimality principle allows to connect classical notions of the Pareto optimality and Nash equilibrium. Lower and upper bounds are obtained for the strong stability radius of the game under parameters perturbations with the assumption that arbitrary Hölder norms are defined in the space of outcomes and criteria space. Game classes with an infinite radius are defined.

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1 Introduction

Rapid development of the various fields of informational technology, economics, social sphere, important part of which is integrity, high complexity and existence of uncertainty factors, requires an adequate development in the corresponding fields of system analysis, management and operations research. One of the main problems arising in this direction is multiobjective decision making in the presence of conflict, uncertainty and risk. An effective tool for modeling decision-making processes is the apparatus of mathematical game theory.

Game-theoretic models target finding classes of outcomes that are rationally coordinated in terms of possible actions and interests of participants (players) or a group of participants (coalitions). For each game in normal form, coalitional and non-coalitional equilibrium concepts (principles of optimality) are used, which usually lead to different game outcomes. In the theory of non-antagonistic games there is no single approach to the development of such concepts. The most famous one is the concept of the Nash equilibrium [13, 14], as well as its various generalizations related to the problems of group choice, which is understood as the reduction of various individual preferences into a single collective preference.

This work implements a parametrization of the equilibrium concept of a finite game in normal form. The parameter of this parametrization is the method of dividing players into coalitions, in which the two extreme cases (a single coalition of players and a set of single-player coalitions) correspond to the Pareto optimal outcome and the Nash equilibrium outcome. Quantitative stability analysis for the set of all efficient (generalized equilibrium) outcomes from the point of view of invariance with respect to changes in the parameters of the game is carried out.

Usually under stability of a multicriteria discrete optimization problem we understand a discrete analog of the Hausdorff upper semicontinuity property [1] of an optimal mapping that defines a choice function, i.e. in our case it is the existence of a neighborhood in the space of game parameters inside which the appearance of new efficient outcomes is not possible. Relaxation of this requirement leads to the stability type which is interpreted as the existence of a neighborhood of initial gains of the game, inside which appearance of new efficient outcomes is possible but for each perturbation there exists at least one efficient outcome of the initial game (not necessarily the same) that remains efficient. Following terminology [10,12], this type of stability is called strong.

In the paper lower and upper bounds of the strong stability radius are found for the game which is optimal for the given partition of players into coalitions under the assumption that arbitrary Hölder's norms are defined in the space of outcomes and criteria space. The classes of all games with infinite strong stability radius are specified. The strong stability radius of the game of finding the Nash set is obtained as a corollary.

Note that analogous quantitative characteristics of the various stability types of multicriteria parameterized problems of game theory and discrete linear programming problems with other principles of optimality, stability types and metrics defined in the space of parameters were obtained in works [2–9].

2 Basic definitions and notations

We consider the main object of study in game theory – finite game of n players in normal form [19], where each player $i \in N_n = \{1, 2, \dots, n\}$, $n \geq 2$, has a set of outcomes $X_i \subset \mathbb{R},\ 2 \leq |X_i| \leq \infty$. The outcome of the game is a realization of the strategies chosen by all the players. This choice is made by the players independently. Let linear payoff functions

$$f_i(x) = C_i x,\ \ i \in N_n,$$

where C_i is the i-th row of a square matrix $C = [c_{ij}] \in \mathbb{R}^{n \times n}$, $x = (x_1, x_2, \dots, x_n)^T \in X_j$, are defined on the set of all outcomes of the game

$$X = \prod_{j \in N_n} X_j \subset \mathbb{R}^n.$$

As a result of the game, which we call the game with matrix C, each player i gains payoff $f_i(x)$ which player tries to maximize using preference relationships.

We assume all players try to maximize own payoffs simultaneously:

$$Cx = (C_1 x, C_2 x, ..., C_n x)^T \to \max_{x \in X}. \tag{1}$$

A non-empty subset $J \subseteq N_n$ is called a coalition of players. For a coalition J and game outcome $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)^T$ we introduce a set

$$V(x^0, J) = \prod_{j \in N_n} V_j(x^0, J)$$

where

$$V_j(x^0, J) = \begin{cases} X_j & \text{if } j \in J, \\ \{x_j^0\} & \text{if } j \in N_n \backslash J. \end{cases}$$

Thus, $V_j(x^0, J)$ is the set of outcomes that are reachable by coalition J from the outcome x^0. It is clear that $V(x^0, N_n) = X$ and $V(x^0, k) = X_k$ for any $x^0, k \in N_n$.

Further we use a binary relation of preference by Pareto [16] $\prec$ in space $\mathbb{R}^k$ of arbitrary dimension $k \in \mathbb{N}$, assuming that for two different vectors $y = (y_1, y_2, \ldots, y_k)^T$ and $y' = (y_1', y_2', \ldots, y_k')^T$ in the space the following formula is valid

$$y \prec y' \;\Leftrightarrow\; y \leq y' \;\&\; y \neq y'.$$

The symbol $\overline{\prec}$, as usual, denotes the negotiation of the relation $\prec$.

Let $s \in N_n$, and let $N_n = \bigcup\limits_{k \in N_s} J_k$ be a partition of the set N_n into s nonempty sets (coalitions), i.e. $J_k \neq \emptyset$, $k \in N_s$, and $p \neq q \Rightarrow J_p \cap J_q = \emptyset$. A set of $(J_1, J_2, ..., J_s)$-efficient outcomes is introduced according to the formula:

$$\begin{aligned} &G(C, J_1, J_2, \ldots, J_s) = \\ &\quad \big\{x \in X \;:\; \forall k \in N_s \quad \forall x' \in V(x, J_k) \quad \big(C_{J_k} x \,\overline{\prec}\, C_{J_k} x'\big)\big\}, \end{aligned} \tag{2}$$

where C_{J_k} is a $|J_k| \times n$ submatrix of matrix C consisting of rows that correspond to players in coalition J_k. For brevity, we denote this set by $G(C)$.

Thus, preference relations between players within each coalition are based on Pareto dominance. Therefore, the set of all N_n-efficient outcomes $G(C, N_n)$ ($s = 1$, i.e. all players are united in one coalition) is Pareto set of game (1) (set of efficient outcomes) [16]:

$$P(C) = \big\{x \in X \;:\; X(x, C) = \emptyset\big\},$$

where

$$X(x, C) = \big\{x' \in X \;:\; Cx \prec Cx'\big\}.$$

Rationality of a cooperative-efficient outcome $x \in P(C)$ is that increase of the payoff of any player is possible only by decreasing the payoff of at least one of the other players.

In the other extreme case, when $s = n$, $G(C, \{1\}, \{2\}, ..., \{n\})$ becomes a set of the Nash equilibria [13, 14]. This set is denoted by $NE(C)$ and defined as follows:

$$NE(C) = \Big\{x \in X \;:\; \nexists k \in N_n \;\; \nexists x' \in X \quad \Big(C_k x < C_k x' \;\&\; x_{N_n \backslash \{k\}} = x'_{N_n \backslash \{k\}}\Big)\Big\},$$

where $x_{N_n \backslash \{k\}}$ is a projection of vector $x \in X$ to the coordinate axis of space $\mathbb{R}^n$ with numbers from the set $N_n \backslash \{k\}$.

It is easy to see that rationality of the Nash equilibrium is that no player can individually deviate from the own equilibrium strategy choice while others keep playing their equilibrium strategies. Strict axioms regarding perfect and common (shared) knowledge are assumed to be fulfilled [15].

Thus, we have just introduced a parametrization of the equilibrium concept for a finite game in normal form. The parameter s of this parameterization is the partitioning of all the players into coalitions $J = (J_1, J_2, ..., J_s)$, in which the two extreme cases (a single coalition of players and a set of n single-player coalitions) correspond to finding the Pareto optimal outcomes $P(C)$ and the Nash equilibrium outcomes $NE(C)$, respectively.

Denoted by $Z(C, J_1, J_2, \dots, J_s)$, the game that consists in finding the set $G(C, J_1, J_2, \dots, J_s)$. Sometimes for brevity, we use the notation $Z(C)$ for this problem.

Without loss of generality, we assume that the elements of partitioning $N_n = \bigcup_{k\in N_s} J_k$ be defined as follows:

$$J_1 = \{1, 2, \dots, t_1\},$$

$$J_2 = \{t_1 + 1, t_1 + 2, \dots, t_2\},$$

$$\dots$$

$$J_s = \{t_{s-1} + 1, t_{s-1} + 2, \dots, n\}.$$

For any $k \in N_s$, let C^k denote a square submatrix of size $|J_k| \times |J_k|$, consisting of those matrix C elements locates at the crossings of rows and columns with numbers J_k, and let $P(C^k)$ be the Pareto set:

$$P(C^k) = \{z \in X_{J_k} \ : \ X(z, C^k) = \emptyset\},$$

where

$$X(z, C^k) = \{z' \in X_{J_k} \ : \ C^k z \prec C^k z'\},$$

of the $|J_k|$-criteria problem $Z(C^k)$.

$$C^k z \to \max_{z\in X_{J_k}},$$

where $z = (z_1, z_2, \dots, z_{|J_k|})^T$, and X_{J_k} is a projection of X onto J_k, i.e.

$$X_{J_k} = \prod_{j\in J_k} X_j \subset \mathbb{R}^{|J_k|}.$$

This problem is called a *partial problem* of the game $Z(C, J_1, J_2, \dots, J_s)$.

Due to the fact that the payoff linear functions $C_i x$, $i \in N_n$, are separable, according to 2, the following equality is valid:

$$G(C, J_1, J_2, \dots, J_s) = \prod_{k=1}^{s} P(C^k). \tag{3}$$

In the definition of $(J_1, J_2, ..., J_s)$-efficiency in the game with matrix $C \in \mathbb{R}^{n\times n}$ only block-diagonal elements $C^1, C^2, \dots, C^s$ matter. Thus, the set of $(J_1, J_2, ..., J_s)$-efficient outcomes of the game $Z(C, J_1, J_2, ..., J_s)$ will be denoted

$$G(\widetilde{C}, J_1, J_2, \dots, J_s),$$

where $\widetilde{C} = \{C^1, C^2, \dots, C^s\}$.

In the space of an arbitrary size $\mathbb{R}^k$, we define Hölder's norm l_p, $p \in [1, \infty]$, i.e. by the norm of the vector $a = (a_1, a_2, ..., a_k)^T \in \mathbb{R}^k$ we mean the number

$$\|a\|_p = \begin{cases} \left(\sum\limits_{j \in N_k} |a_j|^p \right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\left\{ |a_j| \ : \ j \in N_k \right\} & \text{if } p = \infty. \end{cases}$$

The norm of matrix $C \in \mathbb{R}^{k\times k}$ with the rows $C_i, \ i \in N_k$, is defined as the norm of a vector whose components are the norms of the rows of the matrix C. By that, we have

$$\|C\|_{pq} = \left\| (\|C_1\|_p, \|C_2\|_p, \dots, \|C_k\|_p) \right\|_q,$$

where l_q, $q \in [1, \infty]$, is another Hölder's norm, i.e. l_q may differ from l_p in general case.

It is easy to see that for any $p, q \in [1, \infty]$, and for any $i \in N_n$ we have

$$\|C_i\|_p \le \|C\|_{pq}. \tag{4}$$

The norm of the matrix bundle $\widetilde{C} = \{C^1, C^2, \dots, C^s\}, \ C^k \in \mathbb{R}^{|J_k|\times|J_k|}, \ k \in N_s$ is defined as follows:

$$\|\widetilde{C}\|_{\max} = \max\left\{ \|C^k\|_{pq} \ : \ k \in N_s \right\}.$$

Perturbation of the elements of the matrix bundle $\widetilde{C}$ is imposed by adding *perturbing matrix bundle*

$$\tilde{B} = \{B^1, B^2, \dots, B^s\},$$

where $B^k \in \mathbf{R}^{|J_k|\times|J_k|}$ are matrices with rows $B^k_i, \ i \in N_n, \ k \in N_s$. Thus, the set of $(J_1, J_2, ..., J_s)$-efficient outcomes of the perturbed game here and after will be denoted as $G(\widetilde{C} + \tilde{B}, J_1, J_2, \dots, J_s)$.

For an arbitrary number $\varepsilon > 0$, we define a *bundle of perturbing matrices*

$$\Omega^{n\times n}(\varepsilon) = \left\{ \tilde{B} \in \prod_{k=1}^{s} \mathbf{R}^{|J_k|\times|J_k|} \ : \ \|\tilde{B}\|_{\max} < \varepsilon \right\},$$

where

$$\|\tilde{B}\|_{\max} = \max\left\{ \|B^k\|_{pq} \ : \ k \in N_s \right\}.$$

Following the terminology [10,12], the *strong stability radius* (called T_1-stability radius in the terminology of [11,17]) of the game $Z(C, J_1, J_2, \ldots, J_s)$ is the number

$$\rho{=}\rho_{pq}(J_1, J_2, \ldots, J_s){=}\begin{cases} \sup\Xi & \text{if } \ \Xi \neq \emptyset, \\ 0 & \text{if } \ \Xi = \emptyset, \end{cases}$$

where

$$\Xi{=}\Big\{\varepsilon{>}\ 0 : \ \forall \tilde{B} \in \Omega^{\mathrm{n}\times\mathrm{n}}(\varepsilon) \quad \Big(G(\tilde{C}){\cap}G(\tilde{C}+\tilde{B}){\neq}\emptyset\Big)\Big\}.$$

Thus, the strong stability radius determines the limit level of additive perturbations of the elements of the matrix C that preserves optimality of at least one (not necessarily the same) $(J_1, J_2, \ldots, J_s)$-efficient outcome of the original game. It is obvious, when $\overline{G}(\tilde{C}) = \mathrm{X} \setminus \mathrm{G}(\tilde{\mathrm{C}})$ is empty, intersection $G(\tilde{C}){\cap}G(\tilde{C}+\tilde{B}){\neq}\emptyset$ is not empty for any perturbing matrix $B \in \Omega^{\mathrm{n}\times\mathrm{n}}\,(\varepsilon)$ and any $\varepsilon > 0$. Therefore the strong stability radius of the problem is not bounded form above, i.e. $\rho = \infty$. Otherwise $(\overline{G}(\tilde{C}) \neq \emptyset)$ the game $Z(C)$ is called *non-trivial.* By analogy, the partial $|J_k|$-criteria problem $Z(C^k),\ k \in N_s$, is called *non-trivial* if $P(C^k) \neq X_{J_k}$. Thus, according to (3) game $Z(C)$ is non-trivial if and only if among its partial problems there exists at least one non-trivial partial problem.

The strong stability radius of the partial problem $Z(C^k)$ is defined as follows:

$$\rho = \begin{cases} \sup\Theta & if \ \ \Theta \neq \emptyset, \\ 0 & if \ \ \Theta = \emptyset, \end{cases}$$

where

$$\Theta = \{\varepsilon > 0 : \ \forall B^k \in \Omega^{|J_k|\times|J_k|}(\varepsilon) \ \ (P(C^k) \cap P(C^k + B^k) \neq \emptyset)\},$$

$\Omega^{|J_k|\times|J_k|}(\varepsilon) = \{B^k \in \mathbb{R}^{|J_k|\times|J_k|} : \ \|B^k\|_{pq} < \varepsilon\}$ is the set of perturbing matrices of problem $Z(C^k)$.

Obviously, $\rho_{pq}(C^k) = \infty$ if problem $Z(C^k)$ is *trivial*, i.e. $P(C^k) = X_{J_k}$.

The non-trivial $(P(C^k) \neq X_{J_k})$ partial problem $Z(C^k)$ is called *non-degenerate* if the following formula is valid:

$$\exists z^0 \notin P(C^k)\ \exists a \in \mathbb{R}^{|J_k|} \ \ \forall z \in P(C^k) \ \ (a^T(z - z^0) < 0). \tag{5}$$

If the negotiation of the formula is valid, i.e.

$$\forall z \notin P(C^k)\ \forall a \in \mathbb{R}^{|J_k|} \ \ \exists z^0 \in P(C^k) \ \ (a^T(z^0 - z) \geq 0),$$

the problem $Z(C^k)$ is called *degenerate.*

It is easy to see that any scalar ($|J_k| = 1$) non-trivial partial problem $Z(C^k)$ is non-degenerate.

The non-trivial game $Z(C),\ C \in \mathbb{R}^{n\times n}$, is called *non-degenerate* if among its partial problems $Z(C^k),\ k \in N_s$, there exists at least one non-degenerate problem. Non-trivial game $Z(C)$ is called degenerate if all its non-trivial partial problems are degenerate.

3 Auxiliary statements

In the space of an arbitrary dimension $\mathbb{R}^k$, along with the norm $l_p,\ p \in [1, \infty]$, we will use the conjugate norm $l_{p^\star}$, where numbers p and $p^\star$ are connected, as usual, by the equality

$$\frac{1}{p} + \frac{1}{p^\star} = 1,$$

assuming $p^\star = 1$ if $p = \infty$, and $p^\star = \infty$ if $p = 1$. Therefore, we further suppose that the range of variation of the numbers p and $p^\star$ is the closed interval $[1, \infty]$, and the numbers themselves are connected by the above conditions.

Further we use the well-known Hölder's inequality [21]:

$$|a^T b| \leq \|a\|_p \|b\|_{p^\star}, \tag{6}$$

that is true for any two vectors $a = (a_1, a_2, \dots, a_k)^T \in \mathbb{R}^k$ and $b = (b_1, b_2, \dots, b_k)^T \in \mathbb{R}^k$.

Directly from (3), similarly to lemma in [7], the following lemma holds.

Lemma 1. *The outcome $x = (x_1, x_2, \dots, x_n)^T \in X$ is $(J_1, J_2, \dots, J_s)$-efficient, i.e.*

$$x \in G(\tilde{C}, J_1, J_2, \dots, J_s)$$

if and only if for any index $k \in N_s$

$$x_{J_k} \in P(C^k).$$

Hereinafter, x_{J_k} is a projection of vector $x = (x_1, x_2, \dots, x_n)^T$ on coordinate axes of X with coalition numbers J_k.

Further we will use the following notation for the set of non-trivial partial problems of the game $Z(C, J_1, J_2, \dots, J_s)$

$$K(\tilde{C}) = K(\tilde{C}, J_1, J_2, \dots, J_s) = \{k \in N_s : \ P(C^k) \neq X_{J_k}\}.$$

It is easy to see that the following properties are valid.

Property 1. *The game $Z(C, J_1, J_2, \dots, J_s)$ is non-trivial if and only if the set $K(\widetilde{C})$ is non-empty.*

Property 2. *The outcome*

$$x^0 \notin G(\tilde{C}, J_1, J_2, \dots, J_s)$$

if and only if there exists an index $k \in K(\widetilde{C})$ such that

$$x^0_{J_k} \notin P(C^k).$$

Property 3. *If the game $Z(C, J_1, J_2, \dots, J_s)$ is non-trivial, then for any $k \notin K(\widetilde{C})$ we have*

$$P(C^k) = X_{J_k}.$$

Lemma 2. *For any non-degenerate partial problem $Z(C^k)$ there exists non-zero matrix $\hat{C} \in \mathbb{R}^{|J_k|\times|J_k|}$ such that the set $P(C^k) \cap P(\hat{C})$ is empty.*

Proof. According to the definition of the non-degenerate problem $Z(C^k)$ (see (5)) there exist vectors $z^0 \in P(C^k)$ and $a \in \mathbb{R}^{|J_k|}$ such that for each vector $z \in P(C^k)$ the inequality

$$a^T(z - z^0) \leq 0$$

holds.

Therefore, using matrix $\hat{C}$ of size $|J_k| \times |J_k|$ with rows

$$\hat{C}_i = \begin{cases} a & if \quad i = 1, \\ 0^{(n)} & otherwise, \end{cases}$$

where $0^{(n)} = (0, 0, \dots, 0) \in \mathbb{R}^{|J_k|}$, for any vector $z \in P(C^k)$ we have

$$z^0 \in X(z, \hat{C}).$$

In other words, any vector $z \in P(C^k)$ satisfies condition $z \notin P(\hat{C})$. Hence, we have $P(C^k) \cap P(\hat{C}) = \emptyset$. □

Lemma 3. *The strong stability radius $\rho = \rho_{pq}(J_1, J_2, \dots, J_s)$ of the non-degenerate game $Z(C, J_1, J_2, \dots, J_s)$ does not exceed the positive number a if there exists its partial non-degenerate problem $Z(C^r)$ with the radius $\rho_{pq}(C^r) \leq a$.*

Proof. It follows from the conditions of the lemma that for any number $\varepsilon > a$ there exists a perturbing non-zero matrix $B^r = [b_{ij}] \in \Omega^{|J_k|\times|J_k|}(\varepsilon)$ such that

$$P(C^r) \cap P(C^r + B^r) = \emptyset.$$

A perturbing matrix bundle $\tilde{B} = \{B^1, B^2, \dots, B^s\}$ is defined by the rule

$$B^k = \begin{cases} B^r & if \quad k = r, \\ 0^{|J_k|\times|J_k|} & if \quad k = N_s \setminus r, \end{cases}$$

where $0^{|J_k|\times|J_k|}$ is the matrix of size $|J_k| \times |J_k|$ consisting of zeros.

Then we have

$$\|\tilde{B}\|_{max} = \|B^r\|_{pq} < \varepsilon.$$

Therefore, the following formula is valid

$$\varepsilon > a \quad \exists \tilde{B} \in \Omega^{n\times n}(\varepsilon) \quad (G(\tilde{C}) \cap G(\tilde{C} + \tilde{B}) = \emptyset).$$

Hence, $\rho_{pq}(J_1, J_2, \dots, J_s) \leq a$. □

Hereinafter, a^+ is a projection of a vector $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^b$ on a positive orthant, i.e.

$$a^+ = [a]^+ = (a_1^+, a_2^+, \dots, a_b^+),$$

where $+$ implies positive cut of vector a, i.e.

$$a_i^+ = [a_i]^+ = \max\{0, a_i\}, i \in N_n.$$

Lemma 4. *Let numbers $p, q \in [1, \infty]$, $k \in N_s$, φ and vectors z, $z' \in X_{J_k}$ be such that condition*

$$\|[C^k(z - z')]^+\|_q \geq \varphi \, \|z - z'\|_{p^*} > 0$$

holds. Then for any perturbing matrix bundle $\tilde{B} = \{B^1, B^2, \dots, B^s\} \in \Omega^{n\times n}(\varphi)$ we have

$$z' \notin X(z, C^k + B^k).$$

Proof. The proof will be given by contradiction. Assume that there exists a perturbing matrix bundle $\tilde{B} = \{B^1, B^2, \dots, B^s\} \in \Omega^{n\times n}(\varphi)$ such that $z' \in X(z, C^k + B^k)$. Then for any index $i \in J_k$ we derive

$$(C_i^k + B_i^k)z \leq (C_i^k + B_i^k)z',$$

and hence

$$C_i^k(z - z') \leq B_i^k(z' - z).$$

From the last inequality, we continue

$$[C_i^k(z - z')]^+ \leq |B_i^k(z - z')|.$$

Taking into account Hölder's inequality (6), we get

$$[C_i^k(z - z')]^+ \leq \|B_i^k\|_p \|z - z'\|_{p*}.$$

Thus, we conclude

$$\|[C_i^k(z - z')]^+\|_q \leq \|B^k\|_{pq} \|z - z'\|_{p*} \leq \|\tilde{B}\|_{\max} \|z - z'\|_{p*} < \varphi \|z - z'\|_{p*}.$$

The last inequality contradicts the condition of lemma 4. □

For any $x^0 \notin G(\widetilde{C}, J_1, J_2, \dots, J_s)$, we denote

$$K(\tilde{C}, x^0) = \{k \in K(\widetilde{C}) : \; x^0_{J_k} \notin P(C^k)\}.$$

Taking into account properties 1 and 2, we conclude that the next lemma is valid.

Lemma 5. *If $x^0 \notin G(\widetilde{C}, J_1, J_2, \dots, J_s)$, then set $K(\widetilde{C}, x^0)$ is non-empty.*

4 Upper bounds of the radius of a partial problem

Theorem 1. *For the strong stability radius of the non-degenerate problem $Z(C^k)$, $C^k \in \mathbb{R}^{|J_k|\times|J_k|}$, the bound*

$$\rho_{pq}(C^k) \leq \|C^k\|_{pq}$$

is valid.

Proof. Let $\varepsilon > \|C^k\|_{pq}$. Due to the nondegeneracy of the problem $Z(C^k)$ and Lemma 2 there exists a non-zero matrix $\hat{C} \in \mathbb{R}^{|J_k|\times|J_k|}$ such that

$$P(C^k) \cap P(\hat{C}) = \emptyset. \tag{7}$$

Therefore, using as the perturbing matrix $B^k \in \mathbb{R}^{|J_k|\times|J_k|}$ the matrix $\xi\hat{C} - C^k$, where number ξ is defined by the rule

$$0 < \xi < \frac{\varepsilon - \|C^k\|_{pq}}{\|\hat{C}\|_{pq}},$$

we have

$$C^k + B^k = \xi\hat{C},$$

$$\|B^k\|_{pq} = \|\xi\hat{C} - C^k\|_{pq} \leq \xi\|\hat{C}\|_{pq} + \|C^k\|_{pq} < \varepsilon.$$

Thus, due to equality (7) we conclude that for any number $\varepsilon > \|C^k\|_{pq}$ there exists the perturbing matrix $B^k \in \Omega^{|J_k|\times|J_k|}(\varepsilon)$ such that $P(C^k) \cap P(C^k + B^k) = \emptyset$. Hence, $\rho_{pq}(C^k) \leq \|C^k\|_{pq}$. □

Theorem 2. *The strong stability radius $\rho_{pq}(C^k)$ of the degenerate problem $Z(C^k)$, $C^k \in \mathbb{R}^{|J_k|\times|J_k|}$, $|J_k| \geq 2$, equals infinity.*

Proof. Assume the opposite, that the problem has a finite radius. Then there exists a non-zero matrix $\hat{C}^k \in \mathbb{R}^{|J_k|\times|J_k|}$ such that the set $P(C^k) \cap P(\hat{C})$ is empty. Therefore, due to the external stability of the Pareto set (see e.g. [18]), for each vector $z \in P(C^k)$ there exists vector $z'(z) \notin P(C^k)$ such that $z'(z) \in X(z, \hat{C}^k)$. Hence, assuming vector $a \in \mathbb{R}^{|J_k|}$ is equal to the sum of the rows of the matrix $\hat{C}^k$, we conclude that for each vector $z \in P(C^k)$ there exists vector $z'(z) \notin P(C^k)$ such that the inequality is true:

$$a^T(z - z'(z)) < 0. \tag{8}$$

Suppose $z^0 = z'(z^\star)$, where $z^\star$ satisfies the equality

$$a^T z'(z^\star) = \max\{a^T z'(z) : \ z \in P(C^k)\}.$$

Therefore, taking into account inequality (8), we conclude that there exists vector $a \in \mathbb{R}^{|J_k|}$ such that for each vector $z \in P(C^k)$ the inequality

$$a^T(z - z^0) < 0$$

is valid, i.e. formula (5) is true, which means that the problem $Z(C^k)$ is non-degenerate. The contradiction to the degeneracy of the problem proves 2. □

5 Main result

For the non-trivial game $Z(C, J_1, J_2, \dots, J_s)$, $C \in \mathbb{R}^{n\times n}$, $n \geq 2$, $s \in N_s$, and any $p, q \in [1, \infty]$, we define

$$\varphi = \varphi_{pq}(J_1, J_2, \dots, J_s) = \max_{x\in G(\tilde{C})} \min_{k\in K(\tilde{C})} \min_{z\notin P(C^k)} \frac{\|[C^k(x_{J_k} - z)]^+\|_q}{\|x_{J_k} - z\|_{p^*}},$$

$$\psi = \psi_p(J_1, J_2, \dots, J_s) = \min_{x\notin G(\tilde{C})} \max_{k\in K(\tilde{C})} \max_{z\in P(x_{J_k}, C^k)} \min_{i\in J_k} \frac{C_i^k(z - x_{J_k})}{\|z - x_{J_k}\|_{p^*}},$$

$$\|\widetilde{C}\|_{min} = \min\{\|C^k\|_{pq} : \ k \in \hat{K}(\widetilde{C})\},$$

$$\hat{K}(\widetilde{C}) = \hat{K}(\widetilde{C}, J_1, J_2, \dots, J_s) = \{k \in K(\widetilde{C}) : \ Z(C^k) \text{ is a degenerate problem}\},$$

$$P(x_{J_k}, C^k) = X(x_{J_k}, C^k) \cap P(C^k).$$

Theorem 3. *For any $p, q \in [1, \infty]$, $C \in \mathbf{R}^{n\times n}$, $n \geq 2$ and any partition coalition $(J_1, J_2, \dots, J_s)$, $s \in N_n$ the strong stability radius $\rho_{pq}(J_1, J_2, \dots, J_s)$ of the non-trivial game $Z(C, J_1, J_2, \dots, J_s)$ has the following bounds:*

$$0 < \max\{\varphi, \psi\} \leq \rho_{pq}(J_1, J_2, \dots, J_s)$$

$$\begin{cases} \leq \|\widetilde{C}\|_{min} & \textit{if game } Z(C) \textit{ is non-degenerate } (\hat{K}(\widetilde{C}) \neq \emptyset), \\ = \infty & \textit{if game } Z(C) \textit{ is degenerate.} \end{cases}$$

Proof. First of all, due to property 1, the set $K(C)$ is non-empty. Since the formula

$$\forall k \in K(\widetilde{C}) \quad \forall x \in G(\widetilde{C}) \quad \forall z \notin P(C^k) \quad \exists i \in J_k \quad \Big(C_i^k(x_{J_k} - z) > 0\Big)$$

is true, the inequality $\varphi > 0$ is valid.

First, we prove the inequality $\rho \geq \varphi$. Let $\tilde{B} = \{B^1, B^2, \dots, B^s\}$ be the perturbing matrix bundle that belongs to set $\Omega^{n\times n}(\varphi)$. To prove inequality $\rho \geq \varphi$ it is enough to show the existence of the outcome $x^\star \in G(\widetilde{C}) \cap G(\widetilde{C} + \tilde{B})$.

According to the definition of the positive number φ, there exists an outcome $x^0 \in G(\widetilde{C})$ such that for any $k \in K(\widetilde{C})$ and $z \notin P(C^k)$ the relations (taking into account Lemma 1) hold

$$x^0_{J_k} \in P(C^k),$$

$$\|[C^k(x^0_{J_k} - z)]^+\|_q \geq \varphi\|x^0_{J_k} - z\|_{p^\star} > 0.$$

Therefore, according to Lemma 4 the formula is valid

$$\forall k \in K(\tilde{C}) \quad \forall z \notin P(C^k) \quad \forall \tilde{B} \in \Omega^{n\times n}(\varphi) \quad \Big(z \in X(x^0_{J_k}, C^k + B^k)\Big), \tag{9}$$

where $\tilde{B} = (B^1, B^2, \dots, B^s)$.

Next we indicate the way of choosing the necessary outcome $x^\star \in G(\widetilde{C}) \cap G(\widetilde{C} + \tilde{B})$.

If $x^0 \in G(\widetilde{C} + \tilde{B})$, then assume $x^\star = x^0$.

Let $x^0 \notin G(\widetilde{C} + \tilde{B})$. Then due to Lemma 5 $K(\widetilde{C} + \tilde{B}, x^0) \neq \emptyset$. Therefore,

$$x^0_{J_k} \notin P(C^k + B^k), \quad k \in K(\widetilde{C} + \tilde{B}, x^0),$$

$$x^0_{J_k} \in P(C^k + B^k), \quad k \notin K(\widetilde{C} + \tilde{B}, x^0)$$

and assume

$$x^*_{J_k} = x^0_{J_k}, \quad k \notin K(\widetilde{C} + \tilde{B}, x^0). \tag{10}$$

Next, due to the external stability (see e.g. [18]) for each Pareto set $P(C^k + B^k),\ k \in K(widetildeC + \tilde{B}, x^0)$, it is possible to choose vector

$$x^\star_{J_k} \in P(C^k + B^k), \quad k \in K(\widetilde{C} + \tilde{B}, x^0), \tag{11}$$

such that

$$x^\star_{J_k} \in X(x^0_{J_k}, C^k + B^k).$$

Taking into account the proved formula (9), it is easy to see that

$$x^\star_{J_k} \in P(C^k), \quad k \in K(\tilde{C}).$$

Moreover, according to property 3, for any vector $x^\star_{J_k} \in X_{J_k}$ we have

$$x^\star_{J_k} \in P(C^k), \quad k \notin K(\tilde{C}).$$

Thus, according to Lemma 1 $x^\star \in G(\widetilde{C})$. In addition, due to (10) and (11) $x^\star \in G(\widetilde{C} + \tilde{B})$. Hence $x^\star \in G(\widetilde{C}) \cap G(\widetilde{C} + \tilde{B})$ for any perturbing matrix bundle $\tilde{B} \in \Omega^{n\times n}(\varphi)$, i.e. $\rho \geq \varphi$.

Now we prove inequality $\rho \geq \psi$. Assume that $\psi > 0$ (otherwise the inequality is evident).

Let a perturbing matrix bundle

$$\tilde{B} = \{B^1, B^2, \dots, B^s\} \in \Omega^{n\times n}(\psi).$$

Then according to the definition of number ψ for any outcome $x \notin G(\tilde{C})$ there exist $r \in K(\tilde{C})$ and $z^0 \in P(x_{J_r}, C^r)$ such that

$$\frac{C^r_i(z^0 - x_{J_r})}{\|z^0 - x_{J_r}\|_{p^\star}} \geq \psi > \|\tilde{B}\|_{max} \geq \|B^r\|_{pq} \geq \|B^r_i\|_p, \quad i \in J_r.$$

Hence, using the Hölder's inequality (6), we obtain

$$(C^r_i + B^r_i)(z^0 - x_{J_r}) \geq C^r_i(z^0 - x_{J_r}) - \|B^r_i\|_p \|z^0 - x_{J_r}\|_{p^\star} > 0, \quad i \in J_r.$$

It means that

$$x_{J_r} \notin P(C^r + B^r).$$

Therefore, due to Lemma 1 $x \notin G(\widetilde{C} + \tilde{B})$. Summarizing, we conclude that any outcome which is not $(J_1, J_2, \ldots, J_s)$-efficient outcome of the game $Z^n(C)$ remains not efficient in any perturbed game $Z(C + B)$. Thus, the relations are valid

$$\emptyset \neq G(\widetilde{C} + \tilde{B}) \subseteq G(\widetilde{C}).$$

Hence, $G(\widetilde{C}) \cap G(\widetilde{C} + \tilde{B}) \neq \emptyset$ for any perturbing matrix bundle $\tilde{B} \in \Omega^{n\times n}(\psi)$, i.e. $\rho \geq \psi$.

Inequality $\hat{K}(\widetilde{C}) \neq \emptyset$ means that the game $Z(C, J_1, J_2, \ldots, J_s)$ is non-degenerate. Therefore upper bound

$$\rho_{pq}(J_1, J_2, \ldots, J_s) \leq \|\widetilde{C}\|_{min}$$

follows from Theorem 1 and Lemma 3.

Finally, we consider the case when the game $Z(C, J_1, J_2, \ldots, J_s)$ is degenerate. We prove by induction on the number s that the strong stability radius equals infinity.

According to Theorem 2 the strong stability radius of any degenerate partial problem $Z(C^k), k \in N_s$ equals infinity. Therefore, for any matrix $B^s \in \mathbb{R}^{|J_s|\times|J_s|}$ the relation is valid

$$P(C^s) \cap P(C^s + B^s) \neq \emptyset. \tag{12}$$

Further, by induction, we suppose that $\rho_{pq}(J_1, J_2, \ldots, J_{s-1}) = \infty$. It means that for any perturbing matrix bundle $\tilde{B} = \{B^1, B^2, \ldots, B^{s-1}\}, \quad B^k \in \mathbb{R}^{|J_s|\times|J_s|}, k \in N_{s-1}$, we have

$$\prod_{k\in N_{s-1}} P(C^k) \cap \prod_{k\in N_{s-1}} (C^k + B^k) \neq \emptyset. \tag{13}$$

Now suppose that radius $\rho_{pq}(J_1, J_2, \ldots, J_s)$ is finite. Then there exists a perturbing matrix bundle

$$\{B^1, B^2, \ldots, B^s\}, \quad B^k \in \mathbb{R}^{|J_s|\times|J_s|}, k \in N_s,$$

such that

$$\prod_{k\in N_s} P(C^k) \cap \prod_{k\in N_s} (C^k + B^k) = \emptyset,$$

which contradicts inequalities (12) and (13). Hence, the strong stability radius of the degenerate game $Z(C, J_1, J_2, \ldots, J_s)$ equals infinity. □

Now we focus on the strong stability radius $\rho_{pq}(\{1\}, \{2\}, \ldots, \{n\})$ of the non-trivial $(K(\tilde{C}) \neq \emptyset)$ game $Z(C, \{1\}, \{2\}, \ldots, \{n\})$ of finding the Nash set $NE(C)$.

The next corollary follows directly from formula (3).

Corollary 1. *An outcome $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)^T \in X$ of the game with matrix $C \in \mathbb{R}^{n\times n}$ is Nash equilibrium, i.e. $x^0 \in NE(C)$, if and only if the equilibrium strategy of every player $i \in N_n$ is defined as follows*

$$x_i^0 = \begin{cases} \max\{x_i : \ x_i \in X_i\} & if \ c_{ii} > 0, \\ \min\{x_i : \ x_i \in X_i\} & if \ c_{ii} < 0, \\ x_i \in X_i & if \ c_{ii} = 0. \end{cases}$$

Therefore, it is obvious that

$$K(\widetilde{C}) = K(\widetilde{C}, \{1\}, \{2\}, \ldots, \{n\}) = \{k \in N_s : \ c_{kk} \neq 0\}.$$

Since any scalar non-trivial problem is non-degenerate, the game $Z(C, \{1\}, \{2\}, \ldots, \{n\})$ is non-degenerate too. Therefore,

$$\hat{K}(\widetilde{C}) = K(\widetilde{C}) \neq \emptyset. \tag{14}$$

Corollary 2. *For any $p, q \in [1, \infty]$, $C \in \mathbb{R}^{n \times n}$, $n \geq 2$, for the strong stability radius of the non-trivial ($K(\tilde{C}) \neq \emptyset$) game $Z(C, \{1\}, \{2\}, \ldots, \{n\})$ of finding the Nash set $NE(C)$ the formula is valid*

$$\rho^\star = \rho_{pq}(\{1\}, \{2\}, \ldots, \{n\}) = \min\{|c_{kk}| : \ k \in K(\tilde{C})\}.$$

Proof. From Theorem 3 we get the following bounds

$$\varphi^\star \leq \rho^\star \leq \|\tilde{C}\|_{min},$$

where

$$\varphi^\star = \max_{x \in NE(C)} \min_{k \in K(\widetilde{C})} \max_{z \notin P(c_{kk})} \frac{\|[c_{kk}(x_k - z)]^+\|_q}{\|x_k - z\|_{p^\star}},$$

$$\|\tilde{C}\|_{min} = \min\{|c_{kk}| : \ k \in \hat{K}(\tilde{C})\}. \tag{15}$$

Therefore, taking into account Corollary 1, for any $x \in NE(C)$, $k \in K(\tilde{C})$ and $z \notin P(c_{kk})$ the equalities are valid

$$\frac{\|[c_{kk}(x_k - z)]^+\|_q}{\|x_k - z\|_{p^\star}} = \frac{\|c_{kk}(x_k - z)\|_q}{\|x_k - z\|_{p^\star}} = |c_{kk}|.$$

Hence, due to (14) and (15) we conclude

$$\rho^\star = \min\{|c_{kk}| : \ k \in K(\widetilde{C})\}.$$ □

Note that the strong stability radius formula in Corollary 2 implies the attainability of the lower and upper bounds specified in Theorem 3, for the case of the game $Z(C, \{1\}, \{2\}, \ldots, \{n\})$ of finding the Nash set $NE(C)$.

Remark. From Theorem 3 it follows that necessary and sufficient condition of the finite strong stability radius of the game $Z(C, J_1, J_2, \ldots, J_s)$ with $X \neq G(\widetilde{C}, J_1, J_2, \ldots, J_s)$ is the existence of at least one non-degenerate problem among all its non-trivial partial problems $Z(C^1)$, $Z(C^2)$, $\ldots$, $Z(C^s)$. But the existence of the problem $Z(C^k)$, $C^k \in \mathbb{R}^{|J_k|}$, $|J_k| \geq 2$, is equivalent to the consistency of the system of strict linear inequalities

$$y^T(z^0 - z) < 0, \ z \in P(C^k),$$

with unknown vector y for at least one solution $z^0 \notin P(C^k)$. A criterion of the consistency of the system can be found, for instance, in [20].

In conclusion, we give an illustrative example that shows existence of the degenerate problems $Z(C, N_2)$ with condition $P(C) \neq X$ where $C \in \mathbb{R}^{2\times 2}$.

Example. Let the set of outcomes of two players have the form

$$x^\star = (5,2)^T,\ x^1 = (3,1)^T,\ x^2 = (7,3)^T,\ x^3 = (4,4)^T.$$

Thereby two players have strategies $X_1 = \{3,4,5,7\}$ and $X_2 = \{1,2,3,4\}$ but they do not use all of them (to simplify the example).

Let payoff matrix has the form

$$C = \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}.$$

Then we have

$$Cx^\star = (-17,9)^T,\ Cx^1 = (-10,5)^T,\ Cx^2 = (-24,13)^T,$$

$$Cx^3 = (-16,12)^T,\ P(C) = \{x^1, x^2, x^3\},\ x^\star \notin P(C),$$

$$x^\star - x^1 = x^2 - x^\star$$

It is evident that the problem is degenerate and according to theorem 2 its strong stability radius equals infinity.

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