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Morita contexts and closure operators in modules

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Abstract. The relations between the classes of closure operators of two module categories R-Mod and S-Mod are studied in the case when an arbitrary Morita context $(R, {}_RU_S, {}_SV_R, S)$ is given. By the functors $Hom_R(U,\text{-})$ and $Hom_S(V,\text{-})$ two mappings are defined between the closure operators of these categories. Basic properties of these mappings are investigated.

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1 Introduction. Preliminary notions and facts

This work is devoted to the study of closure operators of module categories in the case when we have an arbitrary Morita context $(R, {}_RU_S, {}_SV_R, S)$. The connection between the classes of closure operators $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$ of two module categories R-Mod and S-Mod is investigated. Using the functors $H^U = Hom_R(U,\text{-}) :$ $R\text{-Mod} \rightarrow S\text{-Mod}$ and $H^V = Hom_S(V,\text{-}) : S\text{-Mod} \rightarrow R\text{-Mod}$ two mappings $\mathbb{CO}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{CO}(S)$ are constructed and their principal properties are shown.

Now we remind briefly the basic notions used in continuation. Let R be a ring with unity and R-Mod be the category of unitary left R-modules. The R-morphisms of left R-modules will be written on the right: if $M, M' \in R$-Mod and $f : M \rightarrow M'$ is an R-morphism, then $xf \in M'$ is the image of $x \in M$ in M'. The product (composition) of the R-morphisms $f : M \rightarrow M'$ and $g : M' \rightarrow M''$ is denoted by $f \cdot g : M \rightarrow M''$, where $x(f \cdot g) \overset{\text{def}}{=} (xf)\, g$ for $x \in M$. By $\mathbb{L}(M)$ the lattice of submodules of $M \in R$-Mod is denoted.

The *closure operator* C of the category R-Mod is a function which associates to every pair $N \subseteq M$, where $M \in R$-Mod and $N \in \mathbb{L}(M)$, a submodule $C_M(N)$ of M such that:

(c_1) $N \subseteq C_M(N)$ (*extension*);

(c_2) if $N_1, N_2 \in \mathbb{L}(M)$ and $N_1 \subseteq N_2$, then $C_M(N_1) \subseteq C_M(N_2)$ (*monotony*);

(c_3) for every R-morphism $f : M \rightarrow M'$ and $N \in \mathbb{L}(M)$ the follwing relation is true: $\big[C_M(N)\big] f \subseteq C_{M'}\big(Nf\big)$ (*continuity*) ([3, 4, 6, 7]).

By $\mathbb{CO}(R)$ we denote the class of all closure operators of the category R-Mod.

The *Morita context* $(R,\ {}_RU_S,\ {}_SV_R,\ S)$ consists of two rings R and S, two bimodules ${}_RU_S$ and ${}_SV_R$, and two bimodule morphisms $(,) : U\otimes_S V \to R$ and $[,] : V\otimes_R U \to S$, related by associativity:

$$(u,v)u' = u[v,u'], \qquad [v,u]v' = v(u,v'),$$

where $u, u' \in U$ and $v, v' \in V$ [1, 2, 5, 8–10].

The images of these morphisms $I = (U,V) \in R$ and $J = [V,U] \in S$ are ideals of R and S, respectively, and are called *trace-ideals* of the given Morita context.

Throughout this paper we will consider that an arbitrary Morita context $(R,\ {}_RU_S,\ {}_SV_R,\ S)$ with the associated morphisms $(,)$ and $[,]$ is fixed. Then the following pair of functors is determined:

$$R\text{-Mod} \ \underset{H^V = Hom_S(V,\text{-})}{\overset{H^U = Hom_R(U,\text{-})}{\rightleftarrows}} \ S\text{-Mod},$$

accompanied by two natural transformations:

$$\varphi :\ \mathbb{1}_{R-\mathrm{Mod}} \longrightarrow H^V H^U, \qquad \psi :\ \mathbb{1}_{S-\mathrm{Mod}} \longrightarrow H^U H^V,$$

which are defined as follows.

For every module $X \in R$-Mod we have the R-morphism $\varphi_X : X \to H^V H^U(X)$ which acts by the rule:

$$u(v(x\varphi_X)) \overset{\text{def}}{=\!=} (u,v)x, \tag{1.1}$$

where $x \in X$, $v \in V$ and $u \in U$. Similarly, for every module $Y \in S$-Mod the S-morphism $\psi_Y : Y \to H^U H^V(Y)$ is defined such that:

$$v(u(y\,\psi_Y)) \overset{\text{def}}{=\!=} [v,u]\,y, \tag{1.2}$$

for every $y \in Y$, $u \in U$ and $v \in V$.

These natural transformations φ and ψ are in concordance with the functors H^U and H^V by the following relations:

$$H^U(\varphi_X) \ = \ \psi_{H^U(X)}, \tag{1.3}$$

$$H^V(\psi_Y) \ = \ \varphi_{H^V(Y)}, \tag{1.4}$$

for every $X \in R$-Mod and $Y \in S$-Mod.

2 Mappings between the classes of closure operators

Now we will study the situation described in Section 1: we have a Morita context $(R,\ {}_RU_S,\ {}_SV_R, S)$ and consider the functors $R\text{-Mod} \underset{H^V}{\overset{H^U}{\rightleftarrows}} S\text{-Mod}$ with the natural transformations $\varphi :\ \mathbb{1}_{R-\mathrm{Mod}} \longrightarrow H^V H^U$ and $\psi :\ \mathbb{1}_{S-\mathrm{Mod}} \longrightarrow H^U H^V$. Our purpose is to construct two mappings:

$$\mathbb{CO}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{CO}(S)$$

between the classes of closure operators of the categories R-Mod and S-Mod.

We begin with the mapping $\mathbb{CO}(R) \xrightarrow{(-)^*} \mathbb{CO}(S)$. Let $C \in \mathbb{CO}(R)$. We will define a function C^* in S-Mod which associates to every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod a submodule $C^*_Y(N) \subseteq Y$. With this intention we apply H^V to n and consider in R-Mod the decomposition of the morphism $H^V(n)$ with respect to the given closure operator C:

$$\begin{tikzcd}
H^V(N) \arrow[r, "H^V(n)"] \arrow[d, "\cong"', "\overline{H^V(n)}"] & H^V(Y) \arrow[r, "\pi^n_C", "\text{nat}"'] & H^V(Y)/C_{H^V(Y)}\big(Im\, H^V(n)\big) \\
Im\, H^V(n) \arrow[r, "j^n_C", "\subseteq"'] & C_{H^V(Y)}\big(Im\, H^V(n)\big), \arrow[u, "\cup\!\mid"', "i^n_C"'] &
\end{tikzcd}$$

where $\overline{H^V(n)}$ is the restriction of $H^V(n)$ to its image, j^n_C and i^n_C are the inclusions, π^n_C is the natural epimorphism. Since $H^V(n)$ is a monomorphism, it is obvious that its restriction $\overline{H^V(n)}$ is an isomorphism.

Further, by H^U and ψ $\big($defined by (1.2)$\big)$ we obtain in S-Mod the situation:

$$\begin{tikzcd}
N \arrow[r, "n", "\subseteq"'] \arrow[d, "\psi_N"] & Y \arrow[d, "\psi_Y"] \arrow[dr, dashed] & \\
H^U H^V(N) \arrow[r, "H^U H^V(n)"] \arrow[d, "\cong"', "H^U(\overline{H^V(n)})"] & H^U H^V(Y) \arrow[r, "H^U(\pi^n_C)"] & H^U\big[H^V(Y)/C_{H^V(Y)}\big(Im\, H^V(n)\big)\big] \\
H^U\big(Im\, H^V(n)\big) \arrow[r, "H^U(j^n_C)"] & H^U\big[C_{H^V(Y)}\big(Im\, H^V(n)\big)\big]. \arrow[u, "H^U(i^n_C)"'] &
\end{tikzcd}$$

Definition. For every closure operator $C \in \mathbb{CO}(R)$ we define the function C^* in S-Mod by the following equivalent rules:

$$C^*_Y(N) \overset{\text{def}}{=\!=} \operatorname{Ker}\,[\psi_Y \cdot H^U(\pi^n_C)] \quad \text{or} \quad C^*_Y(N) \overset{\text{def}}{=\!=} [Im\, H^U(i^n_C)]\, \psi_Y^{-1}, \tag{2.1}$$

where $n : N \xrightarrow{\subseteq} Y$ is an arbitrary inclusion of S-Mod.

The equivalence of two forms of representation of a module $C^*_Y(N)$ follows from the left exactness of the functor H^U, which implies the relation: $Im\, H^U(i^n_C) = \operatorname{Ker} H^U(\pi^n_C)$.

In a completely similar manner the inverse mapping $\mathbb{CO}(S) \xrightarrow{(-)^*} \mathbb{CO}(R)$ can be defined. Namely, for every closure operator $D \in \mathbb{CO}(S)$ and every inclusion $m : M \xrightarrow{\subseteq} X$ of R-Mod we consider in S-Mod the decomposition of $H^U(m)$ with respect to the operator D:

$$\begin{tikzcd}
H^U(M) \arrow[r, "H^U(m)"] \arrow[d, "\cong"', "\overline{H^U(m)}"] & H^U(X) \arrow[r, "\pi_D^m", "\text{nat}"'] & H^U(X)/D_{H^U(X)}\big(Im\, H^U(m)\big) \\
Im\, H^U(m) \arrow[r, "j_D^m", "\subseteq"'] & D_{H^U(X)}\big(Im\, H^U(m)\big)\,. \arrow[u, "\cup\!\mid"', "i_D^m"] &
\end{tikzcd}$$

Applying H^V and using φ $\big($defined by (1.1)$\big)$ we obtain in R-Mod the diagram:

$$\begin{tikzcd}
M \arrow[r, "m", "\subseteq"'] \arrow[d, "\varphi_M"] & X \arrow[d, "\varphi_X"] \arrow[dr, dashed] & \\
H^V H^U(M) \arrow[r, "H^V H^U(m)"] \arrow[d, "\cong"', "H^V(\overline{H^U(m)})"] & H^V H^U(X) \arrow[r, "H^V(\pi_D^m)"] & H^V\big[H^U(X)/D_{H^U(X)}\big(Im\, H^U(m)\big)\big] \\
H^V\big(Im\, H^U(m)\big) \arrow[r, "H^V(j_D^m)"] & H^V\big[D_{H^U(X)}\big(Im\, H^U(m)\big)\big]. \arrow[u, "H^V(i_D^m)"'] &
\end{tikzcd}$$

We define the function D^* in R-Mod by the following equivalent rules:

$$D^*_X(M) \overset{\text{def}}{=\!=} \operatorname{Ker}\,[\varphi_X \cdot H^V(\pi_D^m)] \quad \text{or} \quad D^*_X(M) \overset{\text{def}}{=\!=} [Im\, H^V(i_D^m)]\varphi_X^{-1}, \tag{2.2}$$

for every inclusion $m : M \xrightarrow{\subseteq} X$ of R-Mod.

The total symmetry of the investigated situation and of the used methods of construction delivers us from the necessity to repeat for this mapping the results obtained for the previous one. By this reason in continuation we will study mainly the mapping $\mathbb{CO}(R) \xrightarrow{(-)^*} \mathbb{CO}(S)$ and for the inverse mapping we indicate only some affirmations without proofs.

Theorem 2.1. *For every closure operator $C \in \mathbb{CO}(R)$ the function C^* defined by the rule* (2.1) *is a closure operator of the category S-Mod.*

Proof. For the function C^* we will verify the conditions $(c_1) - (c_3)$ of the definition of closure operator.

(c_1) From the construction of C^* and since ψ is natural we have: $n \cdot \psi_Y = \psi_N \cdot H^U H^V(n)$, therefore $n \cdot \psi_Y = \psi_N \cdot H^U\big(\overline{H^V(n)}\big) \cdot H^U(j_C^n) \cdot H^U(i_C^n)$. Hence $Im\,\big(n \cdot \psi_Y\big) \subseteq Im\, H^U(i_C^n)$, i.e. $N\psi_Y \subseteq Im\, H^U(i_C^n)$, which means that $N \subseteq \big[Im\, H^U(i_C^n)\big]\psi_Y^{-1} \overset{\text{def}}{=\!=} C^*_Y(N)$.

(c_2) Let $Y \in S$-Mod and $N_1 \subseteq N_2$, where $N_1, N_2 \in \mathbb{L}(Y)$. We denote the corresponding inclusions as follows:

$$i : N_1 \xrightarrow{\subseteq} N_2, \quad n_1 : N_1 \xrightarrow{\subseteq} Y, \quad n_2 : N_2 \xrightarrow{\subseteq} Y,$$

where $i \cdot n_2 = n_1$. By H^V we obtain in R-mod the situation:

$$
\begin{tikzcd}
H^V(N_1) \arrow[rd, "H^V(n_1)"] \arrow[dd, "H^V(i)"] & \\
& H^V(Y)\,, \\
H^V(N_2) \arrow[ru, "H^V(n_2)"'] &
\end{tikzcd}
$$

whence it follows the inclusion $\kappa : Im\, H^V(n_1) \xrightarrow{\subseteq} Im\, H^V(n_2)$.

Now we will consider in R-mod the decompositions of the morphisms $H^V(n_1)$ and $H^V(n_2)$ with respect to the given operator C:

$$
\begin{tikzcd}
H^V(N_1) \arrow[r, "\overline{H^V(n_1)}"] \arrow[rrrd, "H^V(n_1)"] \arrow[dd, "H^V(i)"] & Im\, H^V(n_1) \arrow[r, "j_C^{n_1}", "\subseteq"'] \arrow[dd, dashed, "\cap\,\kappa"] & C_{H^V(Y)}\big(Im\, H^V(n_1)\big) \arrow[rd, "i_C^{n_1}", "\subseteq"'] \arrow[dd, dashed, "\cap\, l"] & \\
& & & H^V(Y)\,, \\
H^V(N_2) \arrow[r, "\overline{H^V(n_2)}"'] \arrow[rrru, "H^V(n_2)"'] & Im\, H^V(n_2) \arrow[r, "j_C^{n_2}", "\subseteq"'] & C_{H^V(Y)}\big(Im\, H^V(n_2)\big) \arrow[ru, "i_C^{n_2}", "\subseteq"'] &
\end{tikzcd}
$$

where κ implies the inclusion l by the condition (c_2) for C.

Applying H^U to this diagram we obtain in S-Mod the following situation:

$$
\begin{tikzcd}
H^U H^V(N_1) \arrow[r] \arrow[rrd, "H^U H^V(n_1)"] \arrow[dd, "H^U H^V(i)"] & H^U\big[C_{H^V(Y)}\big(Im\, H^V(n_1)\big)\big] \arrow[rd, "H^U(i_C^{n_1})"] \arrow[dd, dashed, "H^U(l)"] & & \\
& & H^U H^V(Y) & Y. \arrow[l, "\psi_Y"'] \\
H^U H^V(N_2) \arrow[r] \arrow[rru, "H^U H^V(n_2)"'] & H^U\big[C_{H^V(Y)}\big(Im\, H^V(n_2)\big)\big] \arrow[ru, "H^U(i_C^{n_2})"'] & &
\end{tikzcd}
$$

Since this diagram commutes, it follows that $Im\, H^U(i_C^{n_1}) \subseteq Im\, H^U(i_C^{n_2})$, therefore $[Im\, H^U(i_C^{n_1})]\psi_Y^{-1} \subseteq [Im\, H^U(i_C^{n_2})]\psi_Y^{-1}$, which by the definition means that $C_Y^*(N_1) \subseteq C_Y^*(N_2)$, proving (c_2).

(c_3) For an arbitrary S-morphism $f : Y \to Y'$ and an inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod now we will show that $[C_Y^*(N)]\, f \subseteq C_{Y'}^*(Nf)$. In S-Mod we have the situation:

$$
\begin{tikzcd}
N \arrow[r, "n", "\subseteq"'] \arrow[d, dashed, "\overline{f}"] & Y \arrow[d, "f"] \\
Nf \arrow[r, dashed, "n'", "\subseteq"'] & Y',
\end{tikzcd}
$$

where $\overline{f}$ is the restriction of f on N and n' is the corresponding inclusion. By H^V we obtain in R-Mod the commutative diagram:

$$\begin{tikzcd}
H^V(N) \arrow[r, "H^V(n)"] \arrow[d, dashed, "H^V(\overline{f})"] & H^V(Y) \arrow[d, "H^V(f)"] \\
H^V(Nf) \arrow[r, dashed, "H^V(n')"] & H^V(Y').
\end{tikzcd}$$

We supplement this diagram considering the decompositions of the morphisms $H^V(n)$ and $H^V(n')$ with respect to the operator C:

$$\begin{tikzcd}
H^V(N) \arrow[d, "\cong", "\overline{H^V(n)}"'] \arrow[rrr, "H^V(n)"] \arrow[dd, dashed, bend right=60, "H^V(\overline{f})"'] & & & \\
Im\, H^V(n) \arrow[r, "j_C^n", "\subseteq"'] \arrow[d, dashed, "{[H^V(f)]'}"] & C_{H^V(Y)}(Im\, H^V(n)) \arrow[r, "i_C^n", "\subseteq"'] \arrow[d, dashed, "{[H^V(f)]''}"] & H^V(Y) \arrow[r, "\pi_C^n", "\mathrm{nat}"'] \arrow[d, "H^V(f)"] & H^V(Y)/C_{H^V(Y)}(Im\, H^V(n)) \arrow[d, dashed, "\pi"] \\
Im\, H^V(n') \arrow[r, "j_C^{n'}"] & C_{H^V(Y)}(Im\, H^V(n')) \arrow[r, "i_C^{n'}", "\subseteq"'] & H^V(Y') \arrow[r, "\pi_C^{n'}", "\mathrm{nat}"'] & H^V(Y')/C_{H^V(Y')}(Im\, H^V(n'))\,. \\
H^V(Nf) \arrow[u, "\cong", "\overline{H^V(n')}"'] \arrow[rru, "H^V(n')"'] & & &
\end{tikzcd}$$

Here the morphism $H^V(f)$ implies $[H^V(f)]'$, which in its turn by the condition (c_3) for C determines the morphism $[H^V(f)]''$. Then in a natural way the morphism π can be defined such that the diagram commutes.

Finally, passing in S-Mod by H^U and using ψ we obtain the diagram:

$$\begin{tikzcd}
C^*_Y(N) \arrow[r, "\subseteq"] \arrow[d, dashed, "f'"] & Y \arrow[r, "\psi_Y"] \arrow[d, "f"] & H^U H^V(Y) \arrow[r, "H^U(\pi_C^n)"] \arrow[d, "H^U H^V(f)"] & H^U\big[H^V(Y)/C_{H^V(Y)}\big(Im\, H^V(n)\big)\big] \arrow[d, dashed, "H^U(\pi)"] \\
C^*_{Y'}(Nf) \arrow[r, "\subseteq"] & Y' \arrow[r, "\psi_{Y'}"] & H^U H^V(Y') \arrow[r, "H^U(\pi_C^{n'})"] & H^U\big[H^V(Y')/C_{H^V(Y')}\big(Im\, H^V(n')\big)\big]\,.
\end{tikzcd}$$

By the definition of C^* and commutativity of this diagram it follows that:

$$\begin{aligned}
C^*_Y(N) \overset{\text{def}}{=\!=} \operatorname{Ker}[\psi_Y \cdot H^U(\pi_C^n)] \;&\subseteq\; \operatorname{Ker}[\psi_Y \cdot H^U(\pi_C^n) \cdot H^U(\pi)] \\
&=\; \operatorname{Ker}[f \cdot \psi_{Y'} \cdot H^U(\pi_C^{n'})]\,.
\end{aligned}$$

Therefore $[C^*_Y(N)]f \subseteq \operatorname{Ker}[\psi_{Y'} \cdot H^U(\pi_C^{n'})] \overset{\text{def}}{=\!=} C^*_{Y'}(Nf)$, which shows the condition (c_3) for C^*, ending the proof. □

By the symmetry for the inverse mapping $\mathbb{CO}(S) \xrightarrow{(-)^*} \mathbb{CO}(R)$ it follows that for every closure operator $D \in \mathbb{CO}(S)$ the function D^* defined by the rule (2.2) is a closure operator of R-Mod.

Further we will illustrate the previous constructions by two particular cases, considering the effect of the studied mappings to the extremal (trivial) closure operators $C = \mathbb{0}_R$ and $C = \mathbb{1}_R$.

1. Let $C = \mathbb{0}_R$, i.e. $C_X(M) = M$ for every inclusion $M \subseteq X$ of R-Mod. By the construction of C^* in this case for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod we have: $C_{H^V(Y)}\big(Im\, H^V(n)\big) = Im\, H^V(n)$, $j^n_C = 1$, $j^n_C \cdot i^n_C = i^n_C : Im\, H^V(n) \xrightarrow{\subseteq} H^V(Y)$ and $Im\, H^U(i^n_C) = Im\, H^U H^V(n)$. Therefore $C^*_Y(N) = [Im\, H^U H^V(n)]\, \psi_Y^{-1}$.

We denote by $D_\circ$ the closure operator of S-Mod defined by the rule:

$$(D_\circ)_Y(N) \stackrel{\text{def}}{=\!=} [Im\, H^U H^V(n)]\, \psi_Y^{-1}$$

for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod. Then from the foregoing it follows that $\mathbb{0}^*_R = D_\circ$ and is clear that $D_\circ$ is the least closure operator of the form C^* for some $C \in \mathbb{CO}(R)$.

2. Let $C = \mathbb{1}_R$, i.e. $C_X(M) = X$ for every inclusion $M \subseteq X$ of R-Mod. Then by the construction of C^* for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod we have $i^n_C = 1$, i.e. $C_{H^V(Y)}\big(Im\, H^V(n)\big) = H^V(Y)$, hence $H^U(i^n_C) = 1$. Therefore $C^*_Y(N) = [H^U H^V(Y)]\, \psi_Y^{-1} = Y$, which means that $C^* = \mathbb{1}_S$.

By these arguments and using the symmetry we have:

Proposition 2.2. 1) $\mathbb{0}_R^{\,*} = D_\circ$, $\mathbb{0}_S^{\,*} = C_\circ$;

2) $\mathbb{1}_R^{\,*} = \mathbb{1}_S$, $\mathbb{1}_S^{\,*} = \mathbb{1}_R$. □

In conclusion of this section we give some remarks on the closure operators defined by the trace-ideals $I = (U, V) \subseteq R$ and $J = [V, U] \subseteq S$ of the studied Morita context $(R, {}_R U_S, {}_S V_R, S)$. The ideals I and J define two preradicals ($r_{(I)}$ in R-Mod and $r_{(J)}$ in S-Mod) by the rules:

$$r_{(I)}(X) \stackrel{\text{def}}{=\!=} \{x \in X \mid Ix = 0\} = \operatorname{Ker} \varphi_X,$$

$$r_{(J)}(Y) \stackrel{\text{def}}{=\!=} \{y \in Y \mid Jy = 0\} = \operatorname{Ker} \psi_Y,$$

where $X \in R$-Mod and $Y \in S$-Mod. These preradicals imply two closure operators (C^I in R-Mod and C^J in S-Mod) defined as follows:

$$C^I_X(M) \stackrel{\text{def}}{=\!=} \{x \in X \mid Ix \subseteq M\},$$

$$C^J_Y(N) \stackrel{\text{def}}{=\!=} \{y \in Y \mid Jy \subseteq N\},$$

for every $M \subseteq X$ of R-Mod and $N \subseteq Y$ of S-Mod.

Now we will show the connection between the closure operators C^I and C^J in the studied situation. Let $m : M \xrightarrow{\subseteq} X$ be an arbitrary inclusion of R-Mod. Using C^I we have the situation:

$$\begin{tikzcd} M \arrow[rr, "m"', "\subseteq"] \arrow[dr, "j^m_{C^I}", "\subseteq"'] & & X. \\ & C^I_X(M) \arrow[ur, "i^m_{C^I}", "\subseteq"'] & \end{tikzcd}$$

We apply the functor H^U and consider the decomposition of the morphism $H^U(m)$ with respect to the operator C^J:

$$\begin{tikzcd}
Im\, H^U(m) \arrow[rr, "j^m_{C^J}", "\subseteq"'] & & C^J_{H^U(X)}\big(Im\, H^U(m)\big) \arrow[dl, "i^m_{C^J}"', "\supseteq"] \arrow[dd, equal] \\
H^U(M) \arrow[u, "\overline{H^U(m)}"', "\cong"] \arrow[r, "H^U(m)"] \arrow[d, "H^U(j^m_{C^I})"] & H^U(X) & \\
H^U\big(C^I_X(M)\big) \arrow[ur, "H^U(i^m_{C^I})"] \arrow[rr, "\overline{H^U(i^m_{C^I})}", "\cong"'] & & Im\, H^U(i^m_{C^I})\,, \arrow[ul, "\supseteq"]
\end{tikzcd}$$

where $\overline{H^U(m)}$ is the restriction of $H^U(m)$ and $\overline{H^U(i^m_{C^I})}$ is the restriction of $H^U(i^m_{C^I})$.

The connection between the operators C^I and C^J(the transition from C^I to C^J) can be expressed as follows.

Proposition 2.3. *For every inclusion $m : M \xrightarrow{\subseteq} X$ of R-Mod the relation $Im\, H^U(i^m_{C^I}) = C^J_{H^U(X)}\big(Im\, H^U(m)\big)$ holds.*

Proof. The left part consists in the following morphisms of $H^U(X)$:

$$\begin{aligned}
Im\, H^U(i^m_{C^I}) &= \{f : U \to X \mid \exists g : U \to C^I_X(M),\ f = g \cdot i^m_{C^I}\} \\
&= \{f : U \to X \mid Uf \subseteq C^I_X(M)\} = \{f : U \to X \mid I(Uf) \subseteq M\} \\
&= \{f : U \to X \mid (U,V)(Uf) = \big(U[V,U]\big)f \subseteq M\} \\
&= \{f : U \to X \mid U\big([V,U]\big)f \subseteq M\}.
\end{aligned}$$

On the other hand, by the definitions we have:

$$\begin{aligned}
C^J_{H^U(X)}\big(Im\, H^U(m)\big) &= \{f : U \to X \mid Jf = [V,U]f \subseteq Im\, H^U(m)\} \\
&= \{f : U \to X \mid U\big([V,U]f\big) \subseteq M\}.
\end{aligned}$$

Comparing the obtained expressions we see the relation of proposition. $\square$

The symmetric relation (which shows the inverse transition from C^J to C^I) also is true: $Im\, H^V(i^n_{C^J}) = C^I_{H^V(Y)}\big(Im\, H^V(n)\big)$ for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod.

3 "Star" mappings and partial order

Further we will investigate the properties of the mappings $\mathbb{CO}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{CO}(S)$ defined in Section 2. We begin with the verification of the behavior of these mappings with respect to the partial orders in the classes $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$. The order relation in $\mathbb{CO}(R)$ is defined in a natural way:

$$C \leq D \Longleftrightarrow C_X(M) \subseteq D_X(M)$$

for every pair $M \subseteq X$ of R-Mod.

Theorem 3.1. *The mapping* $\mathbb{CO}(R) \xrightarrow{(-)^*} \mathbb{CO}(S)$ *defined by the rule* (2.1) *is monotone, i.e.* $C \leq D$ *implies* $C^* \leq D^*$*, where* $C, D \in \mathbb{CO}(R)$*.*

Proof. Let $C, D \in \mathbb{CO}(R)$ and $C \leq D$. Then for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod we have in R-Mod the situation:

$$\begin{tikzcd}
 & C_{H^V(Y)}\big(Im\, H^V(n)\big) \arrow[dr, "i_C^n", "\subseteq"'] \arrow[dd, dashed, "\cap| \, i"] & & H^V(Y)/C_{H^V(Y)}\big(Im\, H^V(n)\big) \arrow[dd, dashed, "\pi"] \\
Im\, H^V(n) \arrow[ur, "j_C^n", "\subseteq"'] \arrow[dr, "j_D^n", "\subseteq"'] & & H^V(Y) \arrow[ur, "\pi_C^n", "nat"'] \arrow[dr, "\pi_D^n", "nat"'] & \\
 & D_{H^V(Y)}\big(Im\, H^V(n)\big) \arrow[ur, dashed, "i_D^n", "\subseteq"'] & & H^V(Y)/D_{H^V(Y)}\big(Im\, H^V(n)\big)\,,
\end{tikzcd}$$

where the inclusion i follows from the relation $C \leq D$, and π is defined by i. Therefore $H^U(\pi_C^n) \cdot H^U(\pi) = H^U(\pi_D^n)$ and so $\operatorname{Ker} H^U(\pi_C^n) \subseteq \operatorname{Ker} H^U(\pi_D^n)$.

Now it is clear that $\operatorname{Ker}[\psi_Y \cdot H^U(\pi_C^n)] \subseteq \operatorname{Ker}[\psi_Y \cdot H^U(\pi_D^n)]$, which means that $C^*_Y(N) \subseteq D^*_Y(N)$ for every inclusion $N \subseteq Y$ of S-Mod, i.e. $C^* \leq D^*$. $\square$

Similarly, the inverse mapping $\mathbb{CO}(S) \xrightarrow{(-)^*} \mathbb{CO}(R)$ also is monotone: $D_1 \leq D_2$ implies ${D_1}^* \leq {D_2}^*$ for $D_1, D_2 \in \mathbb{CO}(S)$.

Concerning the order relations in the classes of closure operators, the following property of the studied mappings deserves to be mentioned.

Theorem 3.2. *For every closure operator* $C \in \mathbb{CO}(R)$ *is true the relation:* $C \leq C^{**}$*.*

Proof. Let $C \in \mathbb{CO}(R)$ and $m : M \xrightarrow{\subseteq} X$ be an arbitrary inclusion of R-Mod. We will compare the R-modules $C_X(M)$ and $C^{**}_X(M)$. With this aim we consider the following inclusions R-Mod:

$$\begin{tikzcd}
M \arrow[r, "j", "\subseteq"'] \arrow[rr, bend left, "m", "\subseteq"'] & C_X(M) \arrow[r, "i", "\subseteq"'] & X\,.
\end{tikzcd}$$

By H^U we obtain in S-Mod the diagram:

$$\begin{tikzcd}
H^U(M) \arrow[r, "H^U(j)"] \arrow[rr, bend left, "H^U(m)"] \arrow[d, "\cong"', "\overline{H^U(m)}"] & H^U\big(C_X(M)\big) \arrow[r, "H^U(i)"] & H^U(X)\,, \\
Im\, H^U(m) \arrow[urr, dashed, "\kappa", "\subseteq"'] & &
\end{tikzcd}$$

where $\overline{H^U(m)}$ is the restriction of $H^U(m)$ and κ is the respective inclusion. To obtain the module $C^{**}_X(M)$ we construct firstly the module $C^*_{H^U(X)}\big(Im\, H^U(m)\big)$, applying the definition of C^* for the inclusion κ (see (2.1)). For that we consider in R-Mod the decomposition of $H^V(\kappa)$ by the operator C, using φ:

$$\begin{tikzcd}
M \arrow[r, "j", "\subseteq"'] \arrow[rr, bend left, "m", "\subseteq"'] \arrow[d, "\varphi_M"] \arrow[ddd, bend right=60, "\overline{\varphi}_M"'] & C_X(M) \arrow[r, "i", "\subseteq"'] \arrow[ddd, dashed, "\varphi'_X"] & X \arrow[ddd, "\varphi_X"] \arrow[dddr, dashed] & \\
H^VH^U(M) \arrow[d, "\cong"', "H^V(\overline{H^U(m)})"] \arrow[ddrr, "H^V(m)"] & & & \\
H^V\big(Im\, H^U(m)\big) \arrow[d, "\cong"', "\overline{H^V(\kappa)}"] \arrow[drr, "H^V(\kappa)"] & & & \\
Im\, H^V(\kappa) \arrow[r, "j^\kappa_C", "\subseteq"'] & C_{H^UH^V(X)}\big(Im\, H^V(\kappa)\big) \arrow[r, "i^\kappa_C", "\subseteq"'] & H^VH^U(X) \arrow[r, "\pi^\kappa_C", "\mathrm{nat}"'] & H^VH^U(X)/B,
\end{tikzcd}$$

where $\overline{\varphi}_M = \varphi_M \cdot H^V(\overline{H^U(m)}) \cdot \overline{H^V(\kappa)}$ and $B = C_{H^VH^U(X)}\big(Im\, H^V(\kappa)\big)$. Using the commutativity and the condition (c_3) for C, we obtain the morphism φ'_X, which preserves the commutativity of diagram.

Further we apply the functor H^U and consider the decomposition of $H^U(m)$ with respect to C^*, obtaining in S-Mod the diagram:

$$\begin{tikzcd}
H^U(M) \arrow[r, "H^U(j)"] \arrow[rrr, bend left, "H^U(m)"] \arrow[d, "\cong"', "\overline{H^U(m)}"] \arrow[ddd, bend right=60, "H^U(\overline{\varphi}_M)"'] & H^U(C_X(M)) \arrow[rr, "H^U(i)"] \arrow[dr, dashed, "\overline{H^U(i)}"] \arrow[ddd, dashed, "H^U(\varphi'_X)"] & & H^U(X) \arrow[r, "\pi^m_{C^*}"] \arrow[ddd, shift right, "H^U(\varphi_X)"'] \arrow[ddd, dashed, shift left, "\psi_{H^U(X)}"] \arrow[dddr, "f"] & H^U(X)/Ker\, f \arrow[d, "\cong"'] \arrow[ddd, bend left=60, "\overline{f}"] \\
Im\, H^U(m) \arrow[rr, "j^m_{C^*}", "\subseteq"'] \arrow[d, "\psi_{Im\, H^U(m)}"] & & A \arrow[ur, dashed, "i^m_{C^*}", "\subseteq"'] & & Im\, f \arrow[dd, "\cap"] \\
H^UH^V\big(Im\, H^U(m)\big) \arrow[d, "\cong"', "H^U(\overline{H^V(\kappa)})"] & & & & \\
H^U(Im\, H^V(\kappa)) \arrow[r, "H^U(j^\kappa_C)"] & H^U[C_{H^VH^U(X)}(Im\, H^V(n))] \arrow[rr, "H^U(i^\kappa_C)"] & & H^UH^VH^U(X) \arrow[r, "H^U(\pi^\kappa_C)"] & H^U[H^VH^U(X)/B],
\end{tikzcd}$$

where we denote $f = \psi_{H^U(X)} \cdot H^U(\pi^\kappa_C)$ and $A = C^*_{H^U(X)}\big(Im\, H^U(m)\big)$.

By the definition of C^* for the inclusion κ we have:

$$C^*_{H^U(X)}\big(Im\, H^U(m)\big) \overset{\mathrm{def}}{=\!=} \mathrm{Ker}\,[\psi_{H^U(X)} \cdot H^U(\pi^\kappa_C)] = \mathrm{Ker}\, f.$$

Using the first isomorphism theorem for the morphism f, we obtain $H^U(X)/\mathrm{Ker}\, f \cong Im\, f$. This isomorphism together with the respective inclusion determines the monomorphism $\overline{f}$, preserving the commutativity of diagram. We remark also that by the relation (1.3) we have $H^U(\varphi_X) = \psi_{H^U(X)}$ (for the dual result (1.4) is used).

Now we will show that the morphism $H^U(i)$ factors through the morphism $i^m_{C^*}$. Indeed, by the construction we have $i^\kappa_C \cdot \pi^\kappa_C = 0$, hence $H^U(i^\kappa_C \cdot \pi^\kappa_C) = H^U(i^\kappa_C) \cdot H^U(\pi^\kappa_C) = 0$. The commutativity of diagram implies the relations:

$$H^U(i) \cdot f = H^U(\varphi'_X) \cdot H^U(i^\kappa_C) \cdot H^U(\pi^\kappa_C) = H^U(\varphi'_X) \cdot 0 = 0\,,$$

therefore $Im\, H^U(i) \subseteq \operatorname{Ker} f \overset{\text{def}}{=\!=} C^*_{H^U(X)}\big(Im\, H^U(m)\big)$. So the morphism $H^U(i)$ can be represented in the form: $H^U(i) = \overline{H^U(i)} \cdot i^m_{C^*}$, preserving the commutativity of diagram.

In continuation we follow the construction of the closure operator C^{**} for the initial inclusion $m : M \xrightarrow{\subseteq} X$, using the module $C^*_{H^U(X)}\big(Im\, H^U(m)\big)$. For that we apply the functor H^V to the necessary part of the previous diagram:

$$\begin{tikzcd}
M \arrow[r, "j", "\subseteq"'] \arrow[rr, bend left, "m", "\subseteq"'] \arrow[d, "\varphi_M"] & C_X(M) \arrow[r, "i", "\subseteq"'] \arrow[d, "\varphi_{C_X(M)}"] & X \arrow[d, "\varphi_X"] \arrow[dr, "g"] & \\
H^VH^U(M) \arrow[r, "H^VH^U(j)"] \arrow[rr, bend left, "H^VH^U(m)"] \arrow[d, "\cong"', "H^V(\overline{H^U(m)})"] & H^VH^U(C_X(M)) \arrow[r, "H^VH^U(i)"] \arrow[d, "H^V(\overline{H^U(i)})"] & H^VH^U(X) \arrow[r, "H^V(\pi^m_{C^*})"] \arrow[d, shift right, "H^VH^U(\varphi_X)"'] \arrow[d, dashed, shift left, "H^V(\psi_{H^U(X)})"] \arrow[dr, "H^V(f)"] & H^V[H^U(X)/A] \arrow[d, "H^V(\overline{f})"] \\
H^V\big(Im\, H^U(m)\big) \arrow[r, "H^V(j^m_{C^*})"] \arrow[urr, dashed, "H^V(\kappa)"] & H^V(A) \arrow[ur, "H^V(i^m_{C^*})"] & H^VH^UH^VH^U(X) \arrow[r, "H^VH^U(\pi^\kappa_C)"] & H^VH^U[H^VH^U(X)/B],
\end{tikzcd}$$

where $g = \varphi_X \cdot H^V(\pi^m_{C^*})$ and by the definition $C^{**}_X(M) = \operatorname{Ker} g$.

We observe that since $\overline{f}$ is a monomorphism, $H^V(\overline{f})$ also is a monomorphism, therefore $\operatorname{Ker}\big(g \cdot H^V(\overline{f})\big) = \operatorname{Ker} g \overset{\text{def}}{=\!=} C^{**}_X(M)$. Since $C^*_{H^U(X)}\big(Im\, H^U(m)\big) \overset{\text{def}}{=\!=} \operatorname{Ker}[\psi_{H^U(X)} \cdot H^U(\pi^\kappa_C)]$, we have the relation $i^m_{C^*} \cdot \psi_{H^U(X)} \cdot H^U(\pi^\kappa_C) = 0$, therefore $H^V[i^m_{C^*} \cdot \psi_{H^U(X)} \cdot H^U(\pi^\kappa_C)] = H^V(i^m_{C^*}) \cdot H^V(\psi_{H^U(X)}) \cdot H^VH^U(\pi^\kappa_C) = 0$.

Using this fact, now from the commutativity of the last diagram we obtain:

$$i \cdot \varphi_X \cdot H^V(\pi^m_{C^*}) \cdot H^V(\overline{f}) = \varphi_{C_X(M)} \cdot H^V(\overline{H^U(i)}) \cdot H^V(i^m_{C^*}) \cdot H^V(\pi^m_{C^*}) \cdot H^V(\overline{f}) =$$

$$= \varphi_{C_X(M)} \cdot H^V(\overline{H^U(i)}) \cdot [H^V(i^m_{C^*}) \cdot H^V(\psi_{H^U(X)}) \cdot H^VH^U(\pi^\kappa_C)] =$$

$$= \varphi_{C_X(M)} \cdot H^V(\overline{H^U(i)}) \cdot 0 = 0.$$

Therefore $C_X(M) \subseteq \operatorname{Ker}[\varphi_X \cdot H^V(\pi^m_{C^*}) \cdot H^V(\overline{f})] = \operatorname{Ker}[\varphi_X \cdot H^V(\pi^m_{C^*})] \overset{\text{def}}{=\!=} C^{**}_X(M)$. So we have $C_X(M) \subseteq C^{**}_X(M)$ for every inclusion $M \subseteq X$, which means that $C \leq C^{**}$. $\square$

The symmetric result also is true: $D \leq D^{**}$ for every operator $D \in \mathbb{CO}(S)$.

4 Intersection of closure operators and "star" mappings

In this section we will study the behavior of the "star" mappings relative to the intersection of closure operators. The intersection in $\mathbb{CO}(R)$ is defined as follows:

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)_X(M) \stackrel{\text{def}}{=\!=} \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_X(M)\big],$$

where $\{C_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}(R)$ and $M \subseteq X$.

Preliminarily we formulate two facts which show the concordance of kernels and preimages of morphisms with the intersection of submodules (see Lemma 4.1).

Let $\{C_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}(R)$ and $n : N \xrightarrow{\subseteq} Y$ be an inclusion of S-Mod. By the definition we have:

$$(C_\alpha)^*_Y(N) \stackrel{\text{def}}{=\!=} \operatorname{Ker}\big[\psi_Y \cdot H^U(\pi^n_{C_\alpha})\big], \tag{4.1}$$

where $\pi^n_{C_\alpha} : H^V(Y) \to H^V(Y)/\big[(C_\alpha)_{H^V(Y)}\big(Im\, H^V(n)\big)\big]$ is the natural epimorphism. Similarly, for the operator $\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha$ we have:

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)^*_Y(N) \stackrel{\text{def}}{=\!=} \operatorname{Ker}\big[\psi_Y \cdot H^U\big(\pi^n_{\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha}\big)\big], \tag{4.2}$$

with the natural epimorphism $\pi^n_{\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha} : H^V(Y) \to H^V(Y)/\big[(\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha)_{H^V(Y)}\big(Im\, H^V(n)\big)\big]$.

Lemma 4.1. *For every family of operators $\{C_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}(R)$ and for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod the following relations are true:*

$$\text{a)} \qquad \operatorname{Ker} H^U\big(\pi^n_{\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha}\big) = \bigcap_{\alpha\in\mathfrak{A}} \big[\operatorname{Ker} H^U(\pi^n_{C_\alpha})\big]; \tag{4.3}$$

$$\text{b)} \qquad \Big[\bigcap_{\alpha\in\mathfrak{A}} \big(\operatorname{Ker} H^U(\pi^n_{C_\alpha})\big)\Big]\psi_Y^{-1} = \bigcap_{\alpha\in\mathfrak{A}} \big[\big(\operatorname{Ker} H^U(\pi^n_{C_\alpha})\big)\psi_Y^{-1}\big]. \tag{4.4}$$

Proof. a) We consider the kernels of morphisms:

$$H^U(\pi^n_{C_\alpha}) \;:\; H^U H^V(Y) \longrightarrow H^U\big[H^V(Y)/(C_\alpha)_{H^V(Y)}\big(Im\, H^V(n)\big)\big],$$

$$H^U\big(\pi^n_{\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha}\big) \;:\; H^U H^V(Y) \longrightarrow H^U\big[H^V(Y)/\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)_{H^V(Y)}\big(Im\, H^V(n)\big)\big].$$

By the definition of H^U we have:

$$\begin{aligned}\operatorname{Ker} H^U(\pi^n_{C_\alpha}) &= \big\{f : U \to H^V(Y) \,\big|\, f \cdot \pi^n_{C_\alpha} = 0\big\} \\ &= \big\{f : U \to H^V(Y) \,\big|\, Im\, f \subseteq (C_\alpha)_{H^V(Y)}\big(Im\, H^V(n)\big)\big\},\end{aligned}$$

$$\begin{aligned} \operatorname{Ker} H^U(\pi^n_{\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha}) &= \big\{ f : U \to H^V(Y) \,\big|\, f \cdot \pi^n_{\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha} = 0 \big\} \\ &= \big\{ f : U \to H^V(Y) \,\big|\, Im\, f \subseteq \big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)_{H^V(Y)} \big(Im\, H^V(n)\big) \\ &= \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_{H^V(Y)}\big(Im\, H^V(n)\big)\big] \big\}. \end{aligned}$$

Therefore

$$\begin{aligned} \bigcap_{\alpha\in\mathfrak{A}} \big[\operatorname{Ker} H^U(\pi^n_{C_\alpha})\big] &= \big\{ f : U \to H^V(Y) \,\big|\, Im\, f \subseteq (C_\alpha)_{H^V(Y)}\big(Im\, H^V(n)\big) \ \forall \alpha \in \mathfrak{A} \big\} \\ &= \big\{ f : U \to H^V(Y) \,\big|\, Im\, f \subseteq \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_{H^V(Y)}\big(Im\, H^V(n)\big)\big] \big\}. \end{aligned}$$

Comparing with the previous relation, now it is clear that $\operatorname{Ker} H^U(\pi^n_{\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha}) = \bigcap\limits_{\alpha\in\mathfrak{A}} \big[\operatorname{Ker} H^U(\pi^n_{C_\alpha})\big]$, i.e. (4.3) is true.

b) In continuation we consider the composition of morphisms:

$$Y \xrightarrow{\psi_Y} H^U H^V(Y) \xrightarrow{H^U(\pi^n_{C_\alpha})} H^U\big[H^V(Y)/(C_\alpha)_{H^V(Y)}\big(Im\, H^V(n)\big)\big],$$

by which the module $(C_\alpha)^*_Y(N) = \operatorname{Ker}[\psi_Y \cdot H^U(\pi^n_{C_\alpha})]$ is defined.

Then we have:

$$y \in \Big[\bigcap_{\alpha\in\mathfrak{A}} \operatorname{Ker} H^U(\pi^n_{C_\alpha})\Big]\psi_Y^{-1} \ \Leftrightarrow\ y\psi_Y \in \bigcap_{\alpha\in\mathfrak{A}} \big[\operatorname{Ker} H^U(\pi^n_{C_\alpha})\big] \ \Leftrightarrow$$

$$\Leftrightarrow\ y\psi_Y \in \operatorname{Ker} H^U(\pi^n_{C_\alpha}) \ \forall \alpha \in \mathfrak{A} \ \Leftrightarrow\ y \in \big[\operatorname{Ker} H^U(\pi^n_{C_\alpha})\big]\psi_Y^{-1} \ \forall \alpha \in \mathfrak{A} \ \Leftrightarrow$$

$$\Leftrightarrow\ y \in \bigcap_{\alpha\in\mathfrak{A}} \big[\big(\operatorname{Ker} H^U(\pi^n_{C_\alpha})\big)\psi_Y^{-1}\big],$$

which proves the relation (4.4). □

Now we can show the concordance of the mapping $\mathbb{CO}(R) \xrightarrow{(-)^*} \mathbb{CO}(S)$ by the intersection of closure operators.

Theorem 4.2. *For every family of closure operators* $\{C_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}(R)$ *the following relation is true:*

$$\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)^* = \bigwedge_{\alpha\in\mathfrak{A}} C_\alpha^*.$$

Proof. For every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod the modules $\big(C_\alpha\big)^*_Y(N)$ and $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha\big)_Y(N)$ are defined by the rules (4.1) and (4.2). From Lemma 4.1 it follows that:

$$\begin{aligned}
\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)^*_Y(N) &\overset{(4.2)}{=\!=} \mathrm{Ker}\,\big[\psi_Y \cdot H^U\big(\pi^n_{\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha}\big)\big] = \big[Ker\, H^U\big(\pi^n_{\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha}\big)\big]\psi_Y^{-1} \\
&\overset{(4.3)}{=\!=} \Big[\bigcap_{\alpha\in\mathfrak{A}} Ker\, H^U(\pi^n_{C_\alpha})\Big]\psi_Y^{-1} \overset{(4.4)}{=\!=} \bigcap_{\alpha\in\mathfrak{A}}\big[\big(Ker\, H^U(\pi^n_{C_\alpha})\big)\psi_Y^{-1}\big] \\
&= \bigcap_{\alpha\in\mathfrak{A}}\big[Ker\big(\psi_Y\cdot H^U(\pi^n_{C_\alpha})\big)\big] \overset{(4.1)}{=\!=} \bigcap_{\alpha\in\mathfrak{A}}\big[\big(C_\alpha\big)^*_Y(N)\big] = \big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha^{\,*}\big)_Y(N),
\end{aligned}$$

which means that $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha\big)^* = \bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha^{\,*}$. □

The symmetric result also holds: $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} D_\alpha\big)^* = \bigwedge\limits_{\alpha\in\mathfrak{A}} D_\alpha^{\,*}$ for every family of operators $\{D_\alpha \,\big|\, \alpha\in\mathfrak{A}\} \subseteq \mathbb{CO}(S)$.

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