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Integral equations in identification of external force and heat source density dynamics

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Abstract. We consider a linear inhomogeneous wave equation and linear inhomogeneous heat equation with initial and boundary conditions. It is assumed that the inhomogeneous terms describing the external force and heat source in the model are decomposed into Fourier series uniformly convergent together with the derivatives up to the second order. In this case, time-dependent expansion coefficients are to be determined. For the purpose of determination of the unknown coefficients, non-local boundary conditions are introduced in accordance with the averaged dynamics required in the model. The nonlocal condition enables the observation of the averaged dynamics of the process. Sufficient conditions are given for the unique classical solution existence. A method for finding the solution of the problem is proposed by reducing to the system of Volterra integral equations of the first kind, which is explicitly constructed in the work. The solution is constructed in explicit form by reduction to Volterra integral equations of the second kind with kernels that admit the construction of the resolvent by means of the Laplace transform. Thus, the work provides a way to solve the identification problem in an analytical form. An illustrative example demonstrating the effectiveness of the proposed approach is given. The statement of the identification problem and the method for solving it allow generalizations also in the case of a system of inhomogeneous equations. The results can be useful in the formulation and solution of the optimization problems of the boundary control process.

Mathematics subject classification: 34A34, 34A12, 35L10, 35L05, 35K05, 43A50, 44A10, 45D05.
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1 Identification of an external force dynamics in the wave equation

We consider an inhomogeneous wave equation describing the forced vibrations of a string under the action of an external force, for zero initial and boundary conditions [15, p. 96].

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + F(x,t), \;\; 0 < x < l, \;\; 0 \le t \le T < \infty. \tag{1}$$

The function $F(x,t) = f(x) + \sum_{j=1}^{N} d_i(x) w_j(t)$ describes the expression for the density (load) of the external force at the point $x \in (0,l)$ at time t [15, p. 26].

It is assumed that the continuous functions $f(x)$, $d_j(x)$ are given for $x \in [0, l]$ and have continuous derivatives in the interval $(0, l)$ up to the second order, continuously continuable to the endpoints of the interval, the third derivatives are piecewise-continuous, and in addition

$$f(0) = f(l) = f^{(2)}(0) = f^{(2)}(l) = d_j(0) = d_j(l) = d_j^{(2)}(0) = d_j^{(2)}(l) = 0,$$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l},\ \ d_j(x) = \sum_{n=1}^{\infty} d_{jn} \sin\frac{n\pi x}{l},\ \ j = 1, ..., N,$$

$$|c_n| = \mathcal{O}\left(\frac{1}{n^4}\right),\ \ |d_{jn}| = \mathcal{O}\left(\frac{1}{n^4}\right).$$

The functions $w_{j(t)}$ characterizing the dynamics of the external force $F(x,t)$, are to be determined. We shall consider the simplest process of oscillations with homogeneous boundary and initial conditions

$$\begin{cases} u(0,t) = u(1,t) = 0, \\ u(x,0) = \frac{\partial u(x,t)}{\partial t}|_{t=0} = 0. \end{cases} \tag{2}$$

For the uniqueness of the definition of the functions $w_{j(t)}$, it is assumed that the averaged dynamics of the oscillations satisfies N nonlocal conditions:

$$\int_0^1 L_i(x) u(x,t) dx = \Delta_i(t),\ \ i = 1, ..., N,\ \ 0 \le t < \infty, \tag{3}$$

where functions $\Delta_i(t)$ and $L_i(x)$ are given, and

$$L_i(x) = \sum_{j=1}^{N} b_{ij} \sin\frac{m_j \pi x}{l},\ \ m_j \in \mathbb{N},$$

$$\Delta_i(0) = \Delta_i^{(1)}(0) = 0.$$

Remark 1. The non-local condition (3) enables the averaged dynamics of oscillations observation. The integral observation conditions were previously used to solve a number of inverse problems for hyperbolic equations [4–6, 9, 10].

Thus, the problem of identifying the functions $w_j(t),\ j = 1, ..., N$, will be solved for modeling the simplest oscillations described by equation (1) with the conditions (2)-(3).

Let us formulate the sufficient conditions for desired continuous and uniquely determined functions $w_j(t)$.

Theorem 1. *Let* $\det[b_{ij}]_{i,j=1}^N \neq 0$, $\det[d_{jm_i}]_{i,j}^N \neq 0$. *Then the problem of identification of functions* $w_j(t),\ j = 1, ..., N$, *has a unique solution in the class of continuous functions. Moreover, these functions are uniquely determined from the system of Volterra integral equations.*

Proof. Following [15, p. 96], the solution $u(x,t)$ of problem (1)-(2) will be constructed in the form of the Fourier series

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)\sin\frac{n\pi x}{l}. \tag{4}$$

Then, in view of the initial conditions corresponding to (2), the functions $u_n(t)$ must satisfy the Cauchy problems

$$\begin{cases} \frac{d^2u_n(t)}{dt^2}+\left(\frac{n\pi a}{l}\right)^2u_n(t)=c_n+\sum_{j=1}^{N}d_{jn}w_j(t), \\ u_n(0)=0,\ u_n'(0)=0,\ \ n=1,2,\dots\ . \end{cases} \tag{5}$$

The solution of the problem (5) is the following

$$u_n(t)=\frac{l}{\pi na}\int_0^t\sin\left(\frac{\pi n}{l}a(t-s)\right)(c_n+\sum_{j=1}^{N}d_{jn}w_j(s))\,ds,\ \ n=1,2,\dots\ . \tag{6}$$

Thus, the expansion (4) of the desired solution $u(x,t)$ is the following

$$u(x,t)=\sum_{n=1}^{\infty}\frac{l}{\pi na}\int_0^t\sin\left(\frac{\pi n}{l}a(t-s)\right)\left(c_n+\sum_{j=1}^{N}d_{jn}w_j(s)\right)ds\sin\frac{\pi nx}{l}+$$

$$+\sum_{n=1}^{\infty}\frac{l}{\pi na}\frac{l}{\pi na}\int_0^t\sin\left(\frac{\pi n}{l}a(t-s)\right)\,dsc_n\sin\frac{\pi nx}{l}, \tag{7}$$

where

$$\int_0^t\sin\left(\frac{\pi n}{l}a(t-s)\right)\,ds=\frac{l}{\pi na}\left(1-\cos\frac{\pi n}{l}at\right).$$

On the other hand, substitution of the expansion (4) into non-local boundary conditions (3) gives the following equalities

$$\int_0^l\sum_{j=1}^{N}b_{ij}\sin\frac{m_j\pi x}{l}\sum_{n=1}^{\infty}\sin\frac{\pi nx}{l}u_n(t)\,dx=\Delta_i(t),\ i=1,\dots,N. \tag{8}$$

Since

$$\int_0^l\sin\frac{m_j\pi x}{l}\sin\frac{\pi nx}{l}\,dx=\begin{cases}0 \text{ at } n\neq m_j,\\ 2/l \text{ at } n=m_j,\end{cases}$$

then the equalities (8) are rewritten as the next system of linear algebraic equations

$$\frac{2}{l}Bv(t)=\Delta(t), \tag{9}$$

where by the hypothesis of the theorem $B=[b_{ij}]_{i,j=1}^{N}$ is an invertible matrix,

$$v(t)=(u_{m_1}(t),\dots,u_{m_N}(t))^T,\ \Delta(t)=(\Delta_1(t),\dots,\Delta_N(t))^T.$$

Thus, the vector function $v(t)$ is determined by the formula

$$(u_{m_1}(t), \ldots, u_{m_N}(t))^T = \frac{l}{2} B^{-1} \Delta(t). \tag{10}$$

Therefore, in order to find the desired functions $w_1(t), \ldots, w_N(t)$ (6) for $n = m_1, \ldots, m_N$ are used. We obtain the following system

$$u_{m_i}(t) = \frac{l}{\pi m_i a} \int_0^t \sin\left(\frac{\pi m_1}{l} a(t-s)\right) \sum_{j=1}^{N} d_{jm_i} w_j(s)\, ds + \tag{11}$$

$$+ (\frac{l}{\pi m_i a})^2 \left(1 - \cos\frac{\pi m_i a}{l} t\right) c_{m_i},$$

where $i = 1, \ldots, N$. According to formula (10) on the left-hand side of system (11) there is a known vector function of the argument t

$$(u_{m_1}(t), \ldots, u_{m_N}(t))^T = \frac{l}{2} B^{-1} \Delta(t). \tag{12}$$

To find the desired functions $w_1(s), \ldots, w_N(s)$ from system (11) we introduce the N auxiliary functions

$$\hat{w}_{m_i}(s) \equiv \sum_{j=1}^{N} d_{jm_i} w_j(s)\, ds, \; i = 1, ..., N. \tag{13}$$

The introduced auxiliary functions $\hat{w}_{m_i}(s)$ (using (11), (12)), must satisfy the following integral equations of the first kind

$$\frac{l}{\pi m_i a} \int_0^t \sin\left(\frac{\pi m_i}{l} a(t-s)\right) \hat{w}_{m_i}(s)\, ds = \tag{14}$$

$$- \left(\frac{l}{\pi m_i a}\right)^2 \left(1 - \cos\frac{\pi m_i a}{l} t\right) c_{m_i} + \delta_i(t),$$

for $i = 1, 2, \ldots, N$, and on the right side we use the notation

$$(\delta_1(t), \ldots, \delta_N(t))^T = \frac{l}{2} B^{-1} (\Delta_1(t), \ldots, \Delta_N(t))^T.$$

Since $\Delta_i(0) = \Delta_i'(0) = 0$, then $\delta_i(0) = \delta_i'(0) = 0$. Therefore, for $t = 0$, the right-hand sides in equations (14) and their derivatives are zeros. Consequently, equations (14) reduced to integral equations of Volterra of the second kind and all continuous functions $\hat{w}_{m_i}(s)$, $i = \overline{1, N}$, are uniquely determined. Substitution into the left-hand side of (13) uniquely determines the desired vector-function $w(t) = (w_1(t), \ldots, w_N(t))^T$ by formula

$$w(t) = D^{-1} \hat{w}(t),$$

where $D = [d_{jm_i}]_{i,j=1}^N$, and $\det D \neq 0$ determined by condition. To complete the proof, it remains to note that, in view of the estimates for the coefficients $|c_n|, |d_{jn}|$, which are valid by virtue of the introduced smoothness requirements for the functions $f(x)$, $d_j(x)$, the Fourier series (7), representing the desired solution $u(x,t)$, converges uniformly together with the derivatives up to the second order. All functions $w_j(t)$ in the expression for the external force are uniquely determined, the solution (7) is classical. The theorem is proved. □

Example 1.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin x w(t), \\ u(0,t) = u(\pi,t) = 0,\ u(x,0) = \frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = 0, \\ \int_0^\pi \sin x\, dx = \frac{\pi}{4} t^2. \end{cases} \tag{15}$$

The function $w_1(t)$ is the desired function. In this example, $N = 1$, $l = \pi$, $L_1(x) = \sin x$, $\Delta(t) = \frac{\pi}{4} t^2$. Therefore, in this example, the equation (14) has the form

$$\int_0^\pi \sin^2 x\, ds \int_0^t \sin(t-s) w_1(s)\, ds = \frac{\pi}{4} t^2.$$

To calculate the function $w_1(t)$, we obtain the integral equation which obviously has a unique solution $w_1(t) = 1 + t^2/2$.

Remark 2. The solution of the integral equations (14) also in the general case is easy to construct explicitly, reducing them to integral Volterra equations of the second kind with kernels of the form $\cos A(t-s)$. In our case $A = \{\dfrac{\pi m_i a}{l}, i = 1, \ldots, N\}$. Indeed, the resolvent $R(t-s)$ of such a kernel can be constructed explicitly from the Laplace transform [2] by the formula $R(t-s) = \dfrac{ae^{a(t-s)} - be^{b(t-s)}}{a-b}$, where a and b are roots of the quadratic equation $p^2 - p + A^2 = 0$.

We note that the above results can be used in the formulation and solution of certain problems of optimization by the boundary control of the oscillation process [3].

Remark 3. The similar result is obtained in the problem of identifying the functions w_{ij}, $i = \overline{1,M}, j = \overline{1,N}$, in the system

$$\frac{\partial^2 u}{\partial t^2} = A^2 \frac{\partial^u}{\partial x^2} + F(x,t), \tag{16}$$

where A is a non-degenerate square matrix of dimension $m \times m$, $u = (u_1(x,t), \ldots, u_m(x,t))^T$, $F = (F_1(x,t), \ldots, F_m(x,t))^T$, $F_i(x,t) = f_i(x) + \sum_{j=1}^N C_{ij}(x) w_{ij}(t)$, $i = \overline{1,M}$, under the local conditions

$$u(0,t) = u(l,t) = 0, \tag{17}$$

$$u(x,0) = \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0, \tag{18}$$

and nonlocal conditions

$$\int_0^l L_{ij}(x) u_j(x,t)\, dx = \Delta_{ij}(t), \tag{19}$$

for $L_{ij}(x) = \sum_{s=1}^{N} b_{ijs} \sin \frac{m_j \pi x}{l}$, $m_j \in \mathbb{N}$. To determine the function $w_{i,j}(t)$ in the external force $F(x,t)$ representation in the system (3.1), we can construct systems of Volterra integral equations. Approximate methods can be used to solve the corresponding integral equations, see, for example, [1, 7, 8].

2 Identification of a heat source dynamics in the heat equation

Consider the heat equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + F(x,t),\ x \in [0, L],\ 0 < t < \infty \tag{20}$$

with boundary conditions

$$u|_{t=0} = 0, \tag{21}$$

$$u|_{x=0} = u|_{x=L} = 0. \tag{22}$$

The function $F(x,t)$ characterizes the density of the heat source (heat release) at the point x at time t. The initial boundary value problem (20)–(22) (see [1]) with a known thermal source occurs in various fields of science and technology, including heat power engineering, hydrology, materials science (see, for example, [15]), etc. If $F(x,t)$ is known, then the solution $u(x,t)$ is constructed in closed form, see [16, p. 214–215]. The problem of restoring the source function and other inverse heat conduction problems has been discussed in recent years, see, for example, [10–12], etc. In these papers, local initial and boundary conditions were used.

Following [10–12], we assume that the function of thermal sources $F(x,t)$ can be represented as the product $f(x)w(t)$. For example, in the case when heat is released as a result of the passage of a current of force I along a rod whose resistance is equal to R, then $F = 0.24 I^2 R$.

We assume that f(x) is a known function from $\mathbb{C}'_{[0,L]}$, $f(0) = f(l) = 0$ and $w(t)$ is the desired one. We are studying the problem of determining the dynamic characteristic of a heat source, i.e. function $w(t)$.

For the purpose of uniqueness of the solution of the problem, we set the desired averaged dynamics of temperature variation in a rod of length L by means of a nonlocal boundary

$$\int_0^L l(x) u(x,t) dx = g(t) \tag{23}$$

considering that

$$l(x) \in \mathbb{C}^{`}_{[0,L]},\ g(t) \in \mathbb{C}'_{[0,T]},\ g(0) = 0$$

are given functions. Then condition (23) is nonlocal boundary condition, and $w(t)$ is the source control. We show that the problem of determining the function $w(t)$ reduces to the solution of the Volterra integral equations (VIE) of the first kind. Indeed, according to [16, p. 183], we can construct a solution of problem (20)–(22) in the form

$$u(x,t) = \int_0^t \int_0^L G(x,\xi,t-\tau)F(\xi,\tau)d\xi d\tau, \tag{24}$$

where

$$G(x,\xi,t-\tau) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-(\frac{\pi n}{L})^2 a^2(t-\tau)} \sin\frac{\pi n}{L}x \sin\frac{\pi n}{L}\xi.$$

To determine the function $w(t)$, we rewrite (24) in the form

$$u(x,t) = \int_0^t \int_0^L G(x,\xi,t-\tau)f(\xi)w(\tau)d\xi d\tau. \tag{25}$$

Substituting the solution (25) into the boundary condition (23), we obtain

$$\int_0^t \iint_0^L l(x)G(x,\xi,t-\tau)f(\xi)dxd\xi w(\tau)d\tau = g(t)$$

Denoting

$$K(t-\tau) := \iint_0^L l(x)G(x,\xi,t-\tau)f(\xi)dxd\xi,$$

we obtain a linear VIE of the first kind with respect to the desired control function $w(t)$:

$$\int_0^t K(t-\tau)w(\tau)d\tau = g(t). \tag{26}$$

Let $f(x)$ and $l(x)$ be differentiable functions, and $\int_0^L f(x)l(x)dx = C \neq 0$. Suppose that the given functions $f(x)$ and $l(x)$ and their derivatives have expansions in uniformly convergent series

$$f^{(i)}(x) = \sum_{n=1}^{\infty} a_n(\sin\frac{\pi n}{L}x)^{(i)},\ l^{(i)}(x) = \sum_{n=1}^{\infty} b_n(\sin\frac{\pi n}{L}x)^{(i)},\ i = \{0,1\}.$$

Then the function $K(t)$ will be a differentiable function, and $K(0) = C \neq 0$. If in this case

$$g(t) \in C'_{(0,+\infty)},\ g(0) = 0,\ \int_0^L f(x)l(x)dx \neq 0,$$

then VIE (26) has a unique continuous solution, which in the general case can be constructed numerically. For a numerical solution, we can use various regularization methods, see [1,8]. For precise $l(x), f(x)$, the form of the kernel in (26) is simplified, which allows us to obtain a solution of (26) in a closed form in a number of cases. Indeed, let

$$f(x) = \sum_{n\in I} a_n \sin\frac{\pi n}{L}x,\ l(x) = \sum_{n\in J} a_n \sin\frac{\pi n}{L}x,$$

where I, J are finite sets of indices from $\mathbf{N}$. We write down the kernel of equation (26) in the form

$$K(t-\tau) = \frac{2}{L}\iint_0^L \sum_{n=1}^{\infty} e^{-(\frac{\pi n}{L})^2 a^2 (t-\tau)} l(x) f(\xi) \sin\frac{\pi n}{L}x \sin\frac{\pi n}{L}\xi dx d\xi.$$

As

$$\int_0^L \sin\frac{\pi n}{L}x \sin\frac{\pi m}{L}x dx = \begin{cases} 0 & \text{if } n \neq m. \\ \frac{L}{2} & \text{n=m.} \end{cases}$$

then in the corresponding VIE we get the kernel in the form of a sum of exponentials:

$$K(t-\tau) = \frac{L}{2}\sum_{n\in I\bigcap J} e^{-(\frac{\pi n}{L})^2} a^2 (t-\tau) a_n b_n.$$

If in this case

$$\frac{L}{2}\sum_{n\in I\bigcap J} a_n b_n \neq 0, \tag{27}$$

$g(t)$ is differentiable, $g(0) = 0$, then the corresponding VIE (26) will have a unique solution. This solution can be obtained numerically or analytically using the Laplace transform. Sometimes, the proposed method makes it possible to obtain a solution in a closed form.

Example 2. Let us consider the case when

$$l(x) = \sin\frac{\pi m x}{L}.$$

Suppose that in this case

$$\int_0^L f(\xi) \sin\frac{\pi m}{L}\xi d\xi \neq 0.$$

In this example, we get the kernel

$$K(t-\tau) = e^{-(\frac{\pi n}{L})^2 a^2 (t-\tau)} \int_0^L f(\xi) \sin\frac{\pi m}{L}\xi d\xi.$$

We introduce the notation $C := \int_0^L f(\xi) \sin\frac{\pi n}{L}\xi d\xi$. Thus, in the case of a nonlocal boundary condition (23) with known function $l(x) = \sin\frac{\pi m x}{L}$ we obtain the simplest VIE of the first kind with respect to $w(t)$:

$$C\int_0^t e^{-(\frac{\pi m}{L})^2 a^2 (t-\tau)} w(\tau) d\tau = g(t). \tag{28}$$

Consequently, when $g(t) \in C'_{[0,+\infty)}$, $g(0) = 0$ in this example, the heat source control function $w(t)$ is defined by the formula:

$$w(t) = \frac{1}{C}\left\{g'(t) + \left(\frac{\pi m}{L}\right)^2 a^2 g(t)\right\}. \tag{29}$$

The corresponding solution (temperature at a point with coordinates x at time t) for a particular component $f(x)$ of a thermal source is determined by formula (25) using solution (29).

If the functions $f(x)$ and $l(x)$ do not have the required smoothness, then the impulse control and the generalized solution of the problem (20)–(23) can be constructed by the approach described in [7].

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