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Properties of finite unrefinable chains of ring topologies for nilpotent rings

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Abstract. Let R be a nilpotent ring and let $(\mathfrak{M}, <)$ be the lattice of all ring topologies or the lattice of all ring topologies in each of which the ring R possesses a basis of neighborhoods of zero consisting of subgroups. If $\tau_0 \prec_{\mathfrak{M}} \tau_1 \prec_{\mathfrak{M}} \ldots \prec_{\mathfrak{M}} \tau_n$ is an unrefinable chain of ring topologies from $\mathfrak{M}$ and $\tau \in \mathfrak{M}$, then $k \leq n$ for any chain $\sup\{\tau, \tau'_0\} = \tau'_1 < \tau'_2 < \ldots < \tau'_k = \sup\{\tau, \tau_n\}$ of topologies from $\mathfrak{M}$.

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1 Introduction

After the problem of the existence of nondiscrete Hausdorff topologies in some infinite rings was solved (see for example, [1, p. 351–390]) it was interesting to study the lattice of all ring topologies and its sublattices.

It was proved (see [7]) that the lattice of all group topologies of an Abelian group is modular.

As properties of finite unrefinable chains were well investigated in any modular lattice (see, for example, Theorem 3.7), then for any Abelian group the properties of finite unrefinable chains in any sublattice of the lattice of all group topologies are well enough investigated.

Although the lattice of all ring topologies may be not modular (see [2]) it is natural to study properties of finite unrefinable chains of ring topologies for some rings.

The present work is a continuation of works [8] and [9] and is devoted to the study of properties of finite unrefinable chains of ring topologies for nilpotent ring.

The basic results of work are Theorem 4.3 and Corollary 4.4 in which properties of a unrefinable chains of ring topologies are proved when passing to the supremum.

For lattices of group topologies similar results are proved in [4].

2 Notations

In this work if another will be not stipulated we shall use the following notations:

2.1. $\mathbb{N}$ is the set of all natural numbers.

2.2. $R(+,\cdot)$ or simply R is a ring and $R(+)$ is the additive group of the ring $R(+,\cdot)$.

2.3. By induction, for any natural number k we shall define the ideal R^k of the ring R as follows:

Put $R^1 = R$ and take as R^{k+1} the subgroup of the group $R(+)$ generated by the set $\bigcup\limits_{i=1}^{k}(R^i \cdot R^{k-i+1})$.

By induction on number n it is easily checked that R^n is an ideal of the ring R.

2.4. If τ_1 and τ_2 are topologies on a set X then we shall consider, that $\tau_1 \leq \tau_2$ if $\tau_1 \subseteq \tau_2$.

2.5. If $(X, <)$ is a partially ordered set, $S \subseteq X$ and $a, b \in X$, then:

– We consider that $a = inf_X S$ if $a \leq x$ for any element $x \in S$ and if $d \in X$ is an element such that $d \leq x$ for all $x \in S$, then $d \leq a$;

– We consider that $b = sup_X S$ if $b \geq x$ for any element $x \in S$ and if $d \in X$ is an element such that $d \geq x$ for all $x \in S$, then $d \geq b$.

3 Definitions and auxiliary results

3.1. Definition (see [3, 5, 6]). A partially ordered set (X, ¡) is called:

– *A lattice* if for any two elements $a, b \in X$ there exist $inf_X\{a, b\}$ and $sup_X\{a, b\}$;

– *A full lattice* if for any nonempty subset $S \subseteq X$ there exist $inf_X S$ and $sup_X S$.

3.2. Remark (see [1], Theorem 1.2.5). Let R be a ring and Ω be a set of subsets of the ring R such that the following conditions are true:

1. $0 \in V$ for any $V \in \Omega$;

2. For any V and U from Ω there exists $W \in \Omega$ such that $W \subseteq V \bigcap U$;

3. For any $V \in \Omega$ there exists $U \in \Omega$ such that $-U \subseteq V$ and $U + U \subseteq V$;

4. For any $V \in \Omega$ there exists $U \in \Omega$ such that $U \cdot U \subseteq V$;

5. For any $V \in \Omega$ and any element $g \in R$ there exists $U \in \Omega$ such that $g \cdot U \subseteq V$ and $U \cdot g \subseteq V$.

Then there exists the unique ring topology τ on the ring R such that Ω is a basis of neighborhoods of zero in the topological ring (R, τ).

Remark 3.3. If I is some ideal of a ring R it is easy to notice that the set $\{I\}$ satisfies conditions 1 – 5 of Remark 3.2 and hence it sets some ring topology in the ring R for which this set is a basis of neighborhoods of zero.

We shall denote this topology by $\tau(I)$.

3.4. Proposition. *For any ring R the following statements are true:*

Statement 3.4.1. *The set $\mathfrak{M}$ of all ring topologies on the ring R is a full lattice;*

Statement 3.4.2. *The set $\mathfrak{G}$ of all ring topologies on the ring R in each of which the topological ring possesses a basis of neighborhoods of zero consisting of subgroups of the group $(R, +)$ is a full lattice.*

Proof. **Statement 3.4.1**. In the beginning we shall show that there exists $sup_{\mathfrak{M}}\mathcal{S}$ for any nonempty subset $\mathcal{S} \subseteq \mathfrak{M}$.

For each ring topology $\tau \in \mathcal{S}$ we shall choose some basis Ω_τ of neighborhoods of zero in the topological ring (R, τ) and also we shall consider the set $\Omega = \bigcup\limits_{\tau\in\mathcal{S}} \Omega_\tau$. If $\widetilde{\Omega}$ is the set of all finite subsets $\Delta \subseteq \Omega$, then for every $\Delta \in \widetilde{\Omega}$ take $\widetilde{W}_\Delta = \bigcap\limits_{V\in\Delta} V$.

It is easy to prove that the set $\Theta = \{\widetilde{W}_\Delta | \Delta \in \widetilde{\Omega}\}$ satisfies conditions $1-5$ of Remark 3.2, and hence, there exists a ring topology $\tau* \in \mathcal{M}$ on the ring R in which the set $\Theta = \{\widetilde{W}_\Delta | \Delta \in \widetilde{\Omega}\}$ is a basis of neighborhoods of zero.

As $\Omega_\tau \subseteq \Theta$ for any topology $\tau \in \mathcal{S}$ then $\tau \leq \tau*$ for any topology $\tau \in \mathcal{S}$.

Let now $\tau' \in \mathfrak{M}$ be a ring topology on ring R such that $\tau \leq \tau'$ for any topology $\tau \in \mathcal{S}$.

Then any subset $V \in \Omega$ is a neighborhood of zero in the topological ring (R, τ'). If $\widetilde{W}_\Delta \in \Theta$, then $\widetilde{W}_\Delta$ is the intersection of finite number of sets from Ω, and hence, it is a neighborhood of zero in the topological ring (R, τ').

Hence $\tau* \leq \tau'$.

So, we have proved that $\tau* = sup_{\mathfrak{M}}\mathcal{S}$.

Now show that there exists $inf_{\mathfrak{M}}\mathcal{S}$ for any nonempty subset $\mathcal{S} \subseteq \mathfrak{M}$.

Consider the set $\mathcal{S}' = \{\tau' \in \mathfrak{M} | \tau' \leq \tau \text{ for all } \tau \in \mathcal{S}\}$. As the set $\mathcal{S}'$ contains the anti-discrete topology then $\mathcal{S}' \neq \emptyset$. Then, as it was proved above, in $\mathfrak{M}$ there exists $\widetilde{\tau} = sup_{\mathfrak{M}}\mathcal{S}'$.

Show that $\widetilde{\tau} = inf_{\mathfrak{M}}\mathcal{S}$.

If $\tau \in \mathcal{S}$, then $\tau' \leq \tau$ for all $\tau' \in \mathcal{S}'$. Then (see 2.5) $\widetilde{\tau} = sup_{\mathfrak{M}}S' \leq \tau$ for all $\tau \in \mathcal{S}$.

Moreover, if $\tau'' \leq \tau$ for all $\tau \in \mathcal{S}$, then $\tau'' \in \mathcal{S}'$, and hence, $\tau'' \leq sup_{\mathfrak{M}}\mathcal{S}' = \widetilde{\tau}$. Then $\widetilde{\tau} = inf_{\mathfrak{M}}\mathcal{S}$.

The Statement 3.4.1 is proved. □

Proof. **Statement 3.4.2**. Let $\emptyset \neq \mathcal{S} \subseteq \mathfrak{G}$ and $\tau* = sup_{\mathfrak{M}}\mathcal{S}$ (see Statement 3.4.1).

As the intersection of any number of subgroups of the group $(R, +)$ is a subgroup, then any subset $\widetilde{W}_\Delta$ is a subgroup of the group $(R, +)$, and hence, $\tau* \in \mathfrak{G}$. As $\mathfrak{G} \subseteq \mathfrak{M}$, then $\tau* = sup_{\mathfrak{G}}\mathcal{S}$.

So, we have proved that there exists $sup_{\mathfrak{G}}\mathcal{S}$

Now show that there exists $inf_{\mathfrak{G}}\mathcal{S}$ for any nonempty subset $\mathcal{S} \subseteq \mathfrak{G}$.

If $\mathcal{S}' = \{\tau' \in \mathfrak{G} | \tau' \leq \tau \text{ for all } \tau \in \mathcal{S}\}$ then, similarly as in the proof of the s Statement 3.3.1 it is proved that $sup_{\mathfrak{G}}\mathcal{S}' = inf_{\mathfrak{G}}\mathcal{S}$.

The Statement 3.4.2 is proved. □

3.5. Definition. (see [3], p. 15) Let $\mathfrak{A}$ be a lattice and $a, b \in \mathfrak{A}$. If $a < b$ and between elements a and b there exist no other elements in the lattice $\mathfrak{A}$ then we shall say that the element b *covers the element* a *in the lattice* $\mathfrak{A}$, and we shall write $a \prec_{\mathfrak{A}} b$.

Notice that if $\mathfrak{A}$ is a sublattice of a lattice $(\mathfrak{B}, <)$ and $a, b \in \mathfrak{A}$, then from $a \prec_{\mathfrak{A}} b$ does not follow that $a \prec_{\mathfrak{B}} b$, but from $a \prec_{\mathfrak{B}} b$ it follows that $a \prec_{\mathfrak{A}} b$.

3.6. Definition. As it is usual (see [3], [6]), a lattice $\mathfrak{A}$ l is called *a modular lattice* if the following condition is true:

If $a, b, c \in \mathfrak{A}$ and $a \leq c$, then $\sup_{\mathfrak{A}}\{a, \inf_{\mathfrak{A}}\{b, c\}\} = \inf_{\mathfrak{A}}\{\sup_{\mathfrak{A}}\{a, b\}, c\}$.

It is easy to notice that any sublattice of a modular lattice is a modular lattice.

3.7. Theorem. *Let $\mathfrak{A}$ be a modular lattice and $a, b \in \mathfrak{A}$. Then the following statements are true:*

Statement 3.7.1. *If $a = a_1 \prec_{\mathfrak{A}} a_2 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} a_n = b$ (i.e. this chain is a unrefinable in the lattice $\mathfrak{A}$) then $k \leq n$ for any chain $a = b_1 < b_2 < \ldots < a_k = b$, and $k = n$ if and only if $a = b_1 \prec_{\mathfrak{A}} b_2 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} a_k = b$ (see [6], pp. 191 and 192);*

Statement 3.7.2 . *If $a, b, c \in \mathfrak{A}$ and $a \prec_{\mathfrak{A}} b$, then $sup_{\mathfrak{A}}\{a,\ c\} \preceq_{\mathfrak{A}} sup_{\mathfrak{A}}\{b,\ c\}$ and $inf_{\mathfrak{A}}\{a,\ c\} \preceq_{\mathfrak{A}} inf_{\mathfrak{A}}\{b,\ c\}$ (see [5], p. 213, theorem 4).*

3.8. Theorem. (see [8], corollary 16) *Let R be a nilpotent ring (i.e. $R^k = \{0\}$ for some natural number k) and let $\mathfrak{A}$ be the lattice $\mathfrak{M}$ of all ring topologies, or it be the lattice $\mathfrak{G}$ of all ring topologies in each of which the ring R possesses a basis of neighborhoods of zero consisting of subgroups.*

If $\tau_0 \prec_{\mathfrak{A}} \tau_1 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau_n$ (i.e. this chain of topologies is a unrefinable chain in $\mathfrak{A}$) and $\tau'_0 < \tau'_1 < \ldots < \tau'_m$ is a chain of ring topologies from $\mathfrak{A}$ such that $\tau_0 = \tau'_0$ and $\tau'_m = \tau_n$, then $m \leq n$.

3.9. Proposition. *Let R be a ring, τ_1 and τ_2 be ring topologies on the ring R. If Ω_1 and Ω_2 are some bases of neighborhoods of zero in topological rings (R, τ_1) and (R, τ_2), accordingly, then the following statements are equivalent:*

Statement 3.9.1. *For any neighborhoods of zero $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$ there exist $V_2 \in \Omega_1$ and $U_2 \in \Omega_2$ such that $V_2 \cdot U_2 \subseteq U_1 + V_1$ and $U_2 \cdot V_2 \subseteq U_1 + V_1$;*

Statement 3.9.2. *The set $\Omega = \{U + V | V \in \Omega_1, U \in \Omega_2\}$ is a basis of neighborhoods of zero of ring topology on the ring R;*

Statement 3.9.3. *The set $\Omega_3 = \{U + V | V \in \Omega_1,\ U \in \Omega_2\}$ is a basis of neighborhoods of zero in the topological ring (R, τ_3), where $\tau_3 = inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\}$.*

Proof. 3.9.1 $\Rightarrow$ 3.9.2.

It is obvious that for the set Ω conditions 1, 2, 3 and 5 of Remark 3.2 are true.

If $W_0 \in \Omega$ then $W_0 = V_0 + U_0$ where $V_0 \in \Omega_1$ and $U_0 \in \Omega_2$. There exist $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$ such that $V_1 + V_1 + V_1 \subseteq V_0$ and $U_1 + U_1 + U_1 \subseteq U_0$ and there exist $V_2 \in \Omega_1$ and $U_2 \in \Omega_2$ such that $V_2 \cdot V_2 \subseteq V_1$ and $U_2 \cdot U_2 \subseteq U_1$.

As the Statement 3.9.1 is executed then we can assume that $V_2 \cdot U_2 \subseteq V_1 + U_1$ and $U_2 \cdot V_2 \subseteq U_1 + V_1$. Then $(U_2+V_2)\cdot(U_2+V_2) \subseteq (U_2 \cdot U_2)+(U_2 \cdot V_2)+(V_2 \cdot U_2)+(V_2 \cdot V_2) \subseteq$

$$U_1 + U_1 + V_1 + U_1 + V_1 + V_1 = U_1 + U_1 + U_1 + V_1 + V_1 + V_1 \subseteq U_0 + V_0 = W_0,$$

i.e. the Statement 3.9.2 is executed, and hence 2.9.1 $\Rightarrow$ 3.9.2. □

Proof. 3.9.2 $\Rightarrow$ 3.9.3.

As the set $\Omega_3 = \{U + V | V \in \Omega_1, U \in \Omega_2\}$ is a basis of neighborhoods of zero of the ring topology τ_3 on the ring R and as $U \subseteq U + V$ and $V \subseteq U + V$ for any $V \in \Omega_1$ and $U \in \Omega_2$, then $\tau_3 \leq \tau_0$ and $\tau_3 \leq \tau_1$, and hence $\tau_3 \leq inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\}$.

Now we shall prove that $\inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\} \leq \tau_3$.

Let $\widetilde{W}_1$ be any neighborhoods of zero of the ring $(R, inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\})$ and let $\widetilde{W}_2$ be such neighborhoods of zero of the ring $(R, inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\})$ then $\widetilde{W}_2 + \widetilde{W}_2 \subseteq \widetilde{W}_1$. As $\tau_1 \leq inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\}$ and $\tau_2 \leq inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\}$, then there exist $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$ such that $V \subseteq \widetilde{W}_2$ and $U \subseteq \widetilde{W}_2$. Then $U + V \subseteq \widetilde{W}_2 + \widetilde{W}_2 \subseteq \widetilde{W}_1$, and as $U + V \in \Omega_3$ then $\inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\} \leq \tau_3$, and hence $\inf_{\mathfrak{M}}\{\tau_1,\ \tau_2\} = \tau_3$, i.e. the statement 3.9.3 is executed, and hence 2.9.2 $\Rightarrow$ 3.9.3 □

Proof. 3.9.3 $\Rightarrow$ 3.9.1. Let $V_1 \in \Omega_1$ and $U_1 \in \Omega_2$. As $\Omega_3 = \{U + V | V \in \Omega_1, U \in \Omega_2\}$ is a basis of neighborhoods of zero of the ring topology τ_3 then there exist $V_2 \in \Omega_1$ and $U_2 \in \Omega_2$ such that $(U_2 + V_2) \cdot (U_2 + V_2) \subseteq U_1 + V_1$. Then

$V_2 \cdot U_2 \subseteq (U_2 + V_2) \cdot (U_2 + V_2) \subseteq U_1 + V_1$ and

$U_2 \cdot V_2 \subseteq (U_2 + V_2) \cdot (U_2 + V_2) \subseteq U_1 + V_1$.

Hence 3.9.3 $\Rightarrow$ 3.9.1. Proposition is completely proved. □

4 The basic results

4.1. Proposition. *Let:*

– *R be a ring;*

– *$\mathfrak{M}$ be the lattice of all ring topologies on the ring R;*

– *$\mathfrak{G}$ be the lattice of all ring topologies on the ring R in which the ring R possesses a basis of neighborhoods of zero consisting of subgroups of the group $(R, +)$;*

– *τ_1 and τ_2 be ring topologies such that topological rings (R, τ_1) and (R, τ_2) possess basis of neighborhoods of zero consisting of subgroups of the group $(R, +)$.*

If for any neighborhood V_0 of zero in the topological ring (R, τ_1) there exist neighborhoods V_1 and U_1 of zero in topological rings (R, τ_1) and (R, τ_2), accordingly, such that $U_1 \cdot V_1 \subseteq V_0$ and $V_1 \cdot U_1 \subseteq V_0$, then $inf_{\mathfrak{M}}\{\tau_1, \tau_2\} = inf_{\mathfrak{G}}\{\tau_1, \tau_2\}$.

Proof. Let Ω_1 and Ω_2 are bases of neighborhoods of zero in topological rings (R, τ_1) and (R, τ_2) accordingly which consist of subgroups.

From Proposition 3.9 it follows that the set $\Omega_3 = \{V + U \mid V \in \Omega_1, U \in \Omega_2\}$ is a basis of neighborhoods of zero in the topological ring $(R, inf_{\mathfrak{M}}\{\tau_1, \tau_2\})$. As $V + U$ is a subgroup of the group $(R, +)$ for any $V \in \Omega_1, U \in \Omega_2$, then $inf_{\mathfrak{M}}\{\tau_1, \tau_2\} \in \mathfrak{G}$, and as $\mathfrak{G} \subseteq \mathfrak{M}$ then $inf_{\mathfrak{M}}\{\tau_1, \tau_2\} = inf_{\mathfrak{G}}\{\tau_1, \tau_2\})$.

Proposition is completely proved. □

4.2. Theorem. *Let R be a ring and let $\mathfrak{A}$ be the lattice $\mathfrak{M}$ of all ring topologies or the lattice $\mathfrak{G}$ of all ring topologies, in each of which the ring R possesses a basis of neighborhoods of zero consisting of subgroups. If τ_0 and τ_1 are ring topologies on R such that $\tau_0 \prec_{\mathfrak{A}} \tau_1$ (the definition of $\prec$ see in 3.5) and $sup_{\mathfrak{A}}\{\tau_0, \tau(R^k)\} = sup_{\mathfrak{A}}\{\tau_1, \tau(R^k)\}$ for some natural number k, then the following statements are true:*

Statement 4.2.1. *If $n = min\{k | sup_{\mathfrak{A}}\{\tau_0, \tau(R^k)\} = sup_{\mathfrak{A}}\{\tau_1, \tau(R^k)\}$ then $\tau_0 = inf_{\mathfrak{A}}\{\tau_1, sup_{\mathfrak{A}}\{\tau_0, \tau(R^{n-1})\}\}$;*

Statement 4.2.2. *For any neighborhood V of zero in the topological ring (R, τ_1) there exist such neighborhoods V_1 and W_1 in topological rings (R, τ_1) and $(R, sup_{\mathfrak{A}}\{\tau_0, \tau(R^{n-1})\}$ (definition of number n see in the formulation of the statement 4.4.1), accordingly, that $W_1 \cdot V_1 \subseteq V$ and $V_1 \cdot W_1 \subseteq V$.*

Statement 4.2.3. *For any neighborhood V of zero in the topological ring (R, τ_1) there exist such neighborhoods V_1 and U in topological rings (R, τ_1) and (R, τ_0) accordingly, that $U \cdot V_1 \subseteq V$ and $V_1 \cdot U \subseteq V$.*

Statement 4.2.4. *If $\tau \in \mathfrak{A}$ then $sup_{\mathfrak{A}}\{\tau, \tau_0\} \preceq_{\mathfrak{M}} sup_{\mathfrak{M}}\{\tau, \tau_1\}$.*

Proof. **Statement 4.2.1**. From definition of the number n it follows that $sup_{\mathfrak{A}}\{\tau_0, \tau(R^{n-1})\} < sup_{\mathfrak{A}}\{\tau_1, \tau(R^{n-1})\}$. Then

$$inf_{\mathfrak{A}}\{\tau_1, sup_{\mathfrak{A}}\{\tau_0, \tau(R^{n-1})\}\} \leq sup_{\mathfrak{A}}\{\tau_0, \tau(R^{n-1})\} < sup_{\mathfrak{A}}\{\tau_1, \tau(R^{n-1})\},$$

and hence, $inf_{\mathfrak{A}}\{\tau_1, sup_{\mathfrak{A}}\{\tau_0, \tau(R^{n-1})\} < \tau_1$.

So, we have received that $\tau_0 \leq inf_{\mathfrak{A}}\{\tau_1, sup_{\mathfrak{A}}\{\tau_0, \tau(R^{n-1})\}\} < \tau_1$.

As $\tau_0 \prec_{\mathfrak{A}} \tau_1$, then $\tau_0 = inf_{\mathfrak{A}}\{\tau_1, sup_{\mathfrak{A}}\{\tau_0, \tau(R^{n-1})\}$.

Statement 4.2.1 is proved. □

Proof. **Statement 4.2.2**. Let V be a neighborhood of zero in the topological ring (R, τ_1) and let V_0 be a neighborhood of zero in the topological ring (R, τ_1) such that $V_0 \cdot V_0 \subseteq V$. From definition of the numbers n it follows that there exists such neighborhood U of zero in topological ring (R, τ_0) that $U \bigcap R^n \subseteq V_0 \bigcap R^n \subseteq V_0$.

There exists a neighborhood U_1 of zero in the topological ring (R, τ_0) and there exists a neighborhood V_0 of zero in the topological ring (R, τ_1) such that $U_1 \cdot U_1 \subseteq U$ and $V_1 \subseteq V_0 \bigcap U_1$. Then from proof of Statement 3.4.1 it follows that $W_1 = U_1 \bigcap R^{n-1}$ will be a neighborhood of zero in the topological ring $(R, sup_{\mathfrak{M}}\{\tau_0, \tau(R^{n-1})\}$.

As $a \cdot g \in R^n \bigcap U$ and $g \cdot a \in R^n \bigcap U$ for any elements $g \in V_1$ and $a \in W_1$, then $g \cdot a \in V_1 \cdot (U \bigcap R^{n-1}) \subseteq V_0 \cdot V_0 \subseteq V$ and $a \cdot g \in (U \bigcap R^{n-1}) \cdot V_1 \subseteq V_0 \cdot V_0 \subseteq V$ for any elements $g \in V_1$ and $a \in W_1$, i.e. $W_1 \cdot V_1 \subseteq V$ and $V_1 \cdot W_1 \subseteq V$.

Statement 4.2.2 is proved. □

Proof. **Statement 4.2.3**. Let V be a neighborhood of zero in the topological ring (R, τ_1). If V_0 is a neighborhood of zero in the topological ring (R, τ_1) such that $V_0 + V_0 \subseteq V$, then from Statement 4.2.2 it follows that there exist neighborhoods

V_2 and W in topological rings (R, τ_1) and $(R, sup_{\mathfrak{A}}\{\tau_0, \tau(R^{n-1})\}$ accordingly, such that $W_1 \cdot V_1 \subseteq V_0$ and $V_1 \cdot W_1 \subseteq V_0$. We can assume that $V_2 \cdot V_2 \subseteq V_1$. Then from Statement 4.2.1 and Proposition 3.9 it follows that $U = V_2 + W_1$ is a neighborhood of zero in the topological ring (R, τ_0) and

$$U \cdot V_2 = (V_2 + W_1) \cdot V_2 = V_2 \cdot V_2 + W_1 \cdot V_2 \subseteq V_1 + V_1 \subseteq V$$

and

$$V_2 \cdot U = V_2 \cdot (V_2 + W_1) = V_2 \cdot V_2 + V_2 \cdot W_1 \subseteq V_1 + V_1 \subseteq V.$$

Statement 4.2.3 is proved. □

Proof. **Statement 4.2.4**. Assume the contrary, i.e. that there exists a ring topology $\tau \in \mathfrak{A}$ such that $sup_{\mathfrak{A}}\{\tau, \tau_1\} > \tau' > sup_{\mathfrak{A}}\{\tau, \tau_0\}$ for some ring topology $\tau' \in \mathfrak{A}$. Then $\tau' > \tau$ and $\tau' > \tau_0$. As $\tau_0 \leq inf\{\tau_1, \tau'\} \leq inf\{\tau_1, sup\{\tau, \tau_1\}\} = \tau_1$ and $\tau_0 \prec_{\mathfrak{A}} \tau_1$ then $\tau_0 = inf\{\tau_1, \tau'\}$ or $inf\{\tau_1, \tau'\} = \tau_1$.

If $\tau_1 = inf\{\tau_1, \tau'\}$ then $\tau' \geq \tau_1$ and as $\tau' > \tau$ then $\tau' \geq sup_{\mathfrak{A}}\{\tau, \tau_1\}$. We have obtained the contradiction with the choice of the topology τ'.

Hence $\tau_0 = inf\{\tau_1, \tau'\}$.

As $\tau' > sup_{\mathfrak{A}}\{\tau, \tau_0\} \geq \tau_0$ then there exists a neighborhood U_0 of zero in the topological ring (R, τ') such that U_0 is not a neighborhood of zero in the topological ring $(R, sup_{\mathfrak{A}}\{\tau, \tau_0\})$.

As $\tau' < sup_{\mathfrak{A}}\{\tau, \tau_1\}$ then there exists a neighborhood V_1 of zero in the topological ring (R, τ_1) and there exist neighborhoods W_0, W_1 and W_2 of zero in the topological ring (R, τ) such that $V_1 \bigcap W_0 \subseteq U_0$, $W_1 + W_1 \subseteq W_0$ and $W_2 - W_2 \subseteq W_1$.

As $\tau' > sup_{\mathfrak{A}}\{\tau, \tau_0\} \geq \tau_0$ then any neighborhood of zero in topological ring (R, τ_0) is a neighborhood of zero in topological ring (R, τ'). Then from Statement 4.2.3 and Proposition 3.9 it follows that $V_1 + W_2$ is a neighborhood of zero in the topological ring (R, τ_0) and hence $(V_1 + W_2) \bigcap W_2$ is a neighborhood of zero in the topological ring $(R, sup_{\mathfrak{A}}\{\tau, \tau_0\})$.

If now $r \in (V_1 + W_2) \bigcap W_2$ then $r = v + w$ where $v \in V_1$ and $w \in W_2$. Then $v = r - w \in W_2 - W_2 \subseteq W_1$, and hence $v \in V_1 \bigcap W_1$. Then

$$r = v + w \in (W_1 + W_1) \bigcap V_1 \subseteq W_0 \bigcap V_1 \subseteq U_0.$$

From the arbitrariness of the element r it follows that $(V_1 + W_2) \bigcap W_2 \subseteq U_0$. As $(V_1 + W_2) \bigcap W_2$ is a neighborhood of zero in the topological ring $(R, sup_{\mathfrak{A}}\{\tau, \tau_0\})$ then U_0 will be a neighborhood of zero in the topological ring $(R, sup_{\mathfrak{A}}\{\tau, \tau_0\})$. We got a contradiction with the choice of the neighborhood U_0.

Theorem is completely proved. □

4.3. Theorem. *Let R be a nilpotent ring and let $\mathfrak{A}$ be the lattice $\mathfrak{M}$ of all ring topologies, or it be the lattice $\mathfrak{G}$ of all ring topologies in each of which the ring R possesses a basis of neighborhoods of zero consisting of subgroups. If $\tau \in \mathfrak{A}$ and $\tau_0 \prec_{\mathfrak{A}} \tau_1 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau_n$ (i.e. this chain of ring topologies is an unrefinable chain in*

$\mathfrak{A}$) then $m \leq n$ for any chain of topologies $\tau'_0 < \tau'_1 < \ldots < \tau'_m$ of ring topologies from $\mathfrak{A}$ such that $sup_{\mathfrak{A}}\{\tau, \tau_0\} = \tau'_0$ and $\tau'_m = sup_{\mathfrak{A}}\{\tau, \tau_n\}$.

Proof. From Theorem 4.2 it follows that the chain

$$sup_{\mathfrak{A}}\{\tau, \tau_0\} \leq sup_{\mathfrak{A}}\{\tau, \tau_1\} \leq \ldots \leq sup_{\mathfrak{A}}\{\tau, \tau_n\}$$

of ring topologies is an unrefinable chain in the lattice $\mathfrak{A}$ and its length is not greater than n. Then from Theorem 3.8 it follows that the length of the chain $\tau'_0 < \tau'_1 < \ldots < \tau'_m$ is not greater tha n.

The Theorem is completely proved. □

4.4. Corollary. *Let:*

– R be a nilpotent ring;

– $\mathfrak{A}$ be the lattice $\mathfrak{M}$ of all ring topologies or it be the lattice $\mathfrak{G}$ of all ring topologies in each of which the ring R possesses basis of neighborhoods of zero consisting of subgroups;

–$\tau \in \mathfrak{A}$;

– $\tau_0 \prec_{\mathfrak{A}} \tau_1 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau_n$ is an unrefinable chain of ring topologies in $\mathfrak{A}$:

$\tau'_0 < \tau'_1 < \ldots < \tau'_m$ is some chain of topologies from $\mathfrak{A}$.

If $sup\{\tau_0, \tau\} = \tau'_0$ and $\tau'_m = sup\{\tau_n, \tau\}$, then $m \leq n$, and $m = n$ if and only if

$\tau'_0 \prec_{\mathfrak{A}} \tau'_1 \prec_{\mathfrak{A}} \ldots \prec_{\mathfrak{A}} \tau'_m$.

Really, as R is a nilpotent ring, then $R^k = \{0\}$ for some natural number k, and hence, $sup\{\tau_0, \tau(R^k)\} = sup\{\tau_0, \tau(0)\} = sup\{\tau_1, \tau(0)\} = sup\{\tau_1, \tau(R^k)\}(\tau_1)|_{R^k}$. Then from Theorem 4.6 the truth of the present corollary follows.

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