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Integrability conditions for a class of cubic differential systems with a bundle of two invariant straight lines and one invariant cubic

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Abstract. We determine conditions for the origin to be a center for a class of cubic differential systems having a bundle of two invariant straight lines and one invariant cubic. We prove that a fine focus $O(0,0)$ is a center if and only if the first three Lyapunov quantities vanish.

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1 Introduction

In this paper we consider the cubic system of differential equations

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{aligned} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y. The origin $O(0,0)$ is a singular point of a center or a focus type for (1), that is, a fine focus. The aim of this paper is to find verifiable conditions for $O(0,0)$ to be a center.

It is known that a singular point $O(0,0)$ is a center for system (1) if and only if it has a holomorphic first integral of the form $F(x,y) = C$ in some neighborhood of $O(0,0)$ [15]. Also, $O(0,0)$ is a center if and only if (1) has a holomorphic integrating factor of the form $\mu = 1 + \sum \mu_j(x,y)$ in some neighborhood of $O(0,0)$ [1].

There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$:

$$\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j.$$

Quantities $L_j,\ j = \overline{1,\infty}$ are polynomials with respect to the coefficients of system (1) called to be the Lyapunov quantities. The origin is a fine focus of order r if $L_1 = L_2 = \ldots = L_{r-1} = 0$ and $L_r \neq 0$.

The origin is a center for (1) if and only if $L_j = 0,\ j = \overline{1,\infty}$. By the Hilbert basis theorem, there is N such that $L_j = 0$ for all j if and only if $L_j = 0$ for all

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$j \le N$. It is only necessary to find a finite number of Lyapunov quantities, though in any given case it is not known a priori how many are required. Thus, the set of points being a center must be an algebraic set, which is called the center variety.

The number N is known for quadratic systems $N = 3$ [2] and for cubic systems with only homogeneous cubic nonlinearities $N = 5$ [22]. If the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of the center has been solved only in some particular cases (see, for example, [3–11, 14, 16–18, 20]).

In this paper we solve the problem of the center for cubic differential system (1) assuming that (1) has two invariant straight lines and one invariant cubic passing through one singular point, i.e. forming a bundle. The paper is organized as follows. In Section 2 we present the known results concerning relation between integrability, invariant algebraic curves and Lyapunov quantities. In Section 3 we find thirty sufficient series of conditions for the existence of a bundle of two invariant straight lines and one invariant conic. In Section 4 we obtain sufficient conditions for the existence of a center and finally we give the proof of the main result: a fine focus $O(0,0)$ is a center for cubic system (1) with a bundle of two invariant straight lines and one invariant cubic if and only if the first three Lyapunov quantities vanish.

2 Algebraic solutions, Lypunov quantities, center

In this paper we study the problem of the center for cubic system (1) assuming that the system has irreducible invariant algebraic curves.

Definition 1. An algebraic curve $\Phi(x, y) = 0$ in $\mathbb{C}^2$ with $\Phi \in \mathbb{C}[x, y]$ is said to be an *invariant algebraic curve* (an algebraic partial integral) of system (1) if

$$\frac{\partial \Phi}{\partial x} P(x, y) + \frac{\partial \Phi}{\partial y} Q(x, y) = \Phi(x, y) K(x, y), \tag{2}$$

for some polynomial $K(x, y) \in \mathbb{C}[x, y]$ called the cofactor of the invariant algebraic curve $\Phi(x, y) = 0$.

If the cubic system (1) has sufficiently many invariant algebraic curves $\Phi_j(x, y) = 0$, $j = 1, \ldots, q$, then in most cases a first integral (an integrating factor) can be constructed in the Darboux form

$$\Phi_1^{\alpha_1} \Phi_2^{\alpha_2} \cdots \Phi_q^{\alpha_q}. \tag{3}$$

Function (3), with $\alpha_j \in \mathbb{C}$ not all zero, is a first integral (an integrating factor) for (1) if and only if

$$\sum_{j=1}^{q} \alpha_j K_j \equiv 0 \quad \Big(\sum_{j=1}^{q} \alpha_j K_j \equiv -\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \Big).$$

If system (1) has a first integral or an integrating factor of the form (3), being $\Phi_j = 0$ invariant algebraic curves of (1), then system (1) is called Darboux integrable. The cubic systems (1) which are Darboux integrable have a center at $O(0,0)$.

The method of Darboux turns out to be very useful and elegant one to prove integrability for some classes of systems depending on parameters. These years, interesting results on algebraic solutions, Lyapunov quantities and Darboux integrability have been obtained (see, for example, [6–11, 16, 19, 20]).

Definition 2. We say that (Φ_k, $k = \overline{1, M}$; $L = N$) is a center sequence for (1) if the existence of M invariant irreducible algebraic curves $\Phi_k(x, y) = 0$ and the vanishing of the Lyapunov quantities L_ν, $\nu = \overline{1, N}$, implies the origin $O(0, 0)$ to be a center for (1).

The problem of center sequences for cubic differential systems with invariant algebraic curves was considered in [5–9]. In these papers, the problem of the center for cubic systems with four invariant straight lines, three invariant straight lines, two invariant straight lines and one invariant conic was completely solved. The main results of these works are summarized in the following theorem.

Theorem 1. *($a_jx + b_jy + c_j$, $j = \overline{1, 4}$; $L = 2$), ($a_jx + b_jy + c_j$, $j = \overline{1, 3}$; $L = 7$) and ($a_jx + b_jy + c_j$, $j = 1, 2$, $a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0$; $L = 4$) are center sequences for the cubic system (1).*

The problem of the center for cubic system (1) having two parallel invariant straight lines and one invariant cubic was solved in [11]. In this paper, we have obtained the following result.

Theorem 2. *($l_j = a_jx+b_jy+c_j$, $j = 1, 2$, $l_1||l_2$, $x^2+y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3 = 0$; $L = 2$) is a center sequence for the cubic system (1).*

The problem of center sequences for some classes of cubic systems (1) having bundles of invariant algebraic curves was considered in [8, 9]. The main results of these papers are summarized in the following theorem.

Theorem 3. *($1+a_jx-y$, $j = 1, 2, 3$; $L = 5$), ($1+a_jx-y$, $j = 1, 2$, $a_{20}x^2+a_{11}xy+a_{10}x + (a_{02}y - 1)(y - 1) = 0$; $L = 4$) are center sequences for the cubic system (1).*

In the present paper, we shall prove that ($l_j = 1 + a_jx - y$, $j = 1, 2$, $l_1 \cap l_2 \cap \Phi = (0, 1)$; $L = 3$), where $\Phi = x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3$ is an irreducible invariant cubic, is a center sequence for the cubic system (1).

3 Conditions for the existence of a bundle of two invariant straight lines and one invariant cubic

Let the cubic system (1) have two invariant straight lines l_1, l_2 intersecting at a real singular point (x_0, y_0). By rotating the system of coordinates ($x \to x\cos\varphi - y\sin\varphi, y \to x\sin\varphi+y\cos\varphi$) and rescaling the axes of coordinates ($x \to \alpha x$, $y \to \alpha y$), we obtain $l_1 \cap l_2 = (0, 1)$. In this case the invariant straight lines can be written as

$$l_j \equiv 1 + a_jx - y = 0,\ a_j \in \mathbb{C},\ j = 1, 2;\ a_2 - a_1 \neq 0. \tag{4}$$

Cubic systems with at most two invariant straight lines, including the line at infinity, of the maximal multiplicity were investigated in [21]. Center conditions for a cubic system (1) with two distinct invariant straight lines by using the method of Darboux integrability are obtained in [10]. According to [10] the straight lines (4) are invariant for (1) if and only if the following coefficient conditions are satisfied:

$$\begin{aligned} &k=(a-1)(a_1+a_2)+g,\ \ l=-b,\ \ s=(1-a)a_1a_2,\\ &m=-a_1^2-a_1a_2-a_2^2+c(a_1+a_2)-a+d+2,\ \ r=-f-1,\\ &n=a_1a_2(-f-2)-(d+1),\ \ p=(f+2)(a_1+a_2)+b-c,\\ &q=(a_1+a_2-c)a_1a_2-g,\ \ (a-1)^2+(f+2)^2\neq 0. \end{aligned}$$

In this case the cubic system (1) looks:

$$\begin{aligned} \dot{x}=\ & y+ax^2+cxy+fy^2+[(a-1)(a_1+a_2)+g]x^3+\\ &+[d+2-a-a_1^2-(a_1+a_2)(a_2-c)]x^2y+\\ &+[(f+2)(a_1+a_2)+b-c]xy^2-(f+1)y^3\equiv P(x,y),\\ \dot{y}=\ & -x-gx^2-dxy-by^2+(a-1)a_1a_2x^3+[g+a_1a_2(c-\\ &-a_1-a_2)]x^2y+[(f+2)a_1a_2+d+1]xy^2+by^3\equiv Q(x,y). \end{aligned} \tag{5}$$

Next for cubic system (5) we find conditions for the existence of one invariant cubic curve passing through the same singular point $(0,1)$, i.e. forming a bundle. Let the cubic curve be given by the equation

$$\Phi(x,y)\equiv x^2+y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3=0 \tag{6}$$

with $(a_{30},a_{21},a_{12},a_{03})\neq 0$ and $a_{30},a_{21},a_{12},a_{03}\in\mathbb{R}$.

In order the cubic curve (6) pass through a singular point $(0,1)$ and form a bundle with the invariant straight lines (4), we shall assume $a_{03}=-1$.

By Definition 1, the cubic curve (6) is an invariant cubic curve for system (5) if there exist numbers c_{20}, c_{11}, c_{02}, c_{10}, $c_{01}\in\mathbb{R}$ such that

$$P(x,y)\frac{\partial\Phi}{\partial x}+Q(x,y)\frac{\partial\Phi}{\partial y}\equiv\Phi(x,y)(c_{20}x^2+c_{11}xy+c_{02}y^2+c_{10}x+c_{01}y). \tag{7}$$

Identifying the coefficients of the monomials x^iy^j in (7), we reduce this identity to a system of fifteen equations $\{F_{ij}=0\}$ for the unknowns $a_{30},a_{21},a_{12},c_{20},c_{11},c_{02}$, c_{10},c_{01}. We find that

$$\begin{aligned} c_{20}\ &=[2(f+1)a_{12}(a_{12}^2+3a_{21})-2((f+2)(a_1+a_2)-c)(a_{12}^2+2a_{21})+\\ &+a_{12}(2a-11+2(a_1+a_2)^2+2a_1a_2(f+1)-2c(a_1+a_2))+\\ &+3a_{30}(2f+5)+6(b+c(a_1a_2+1)-a_1a_2(a_1+a_2))]/2,\\ c_{11}\ &=[2(f+1)a_{12}^2-2a_{12}((f+2)(a_1+a_2)-c)+a_{21}(4f+13)+\\ &+3(5-2a+2f+2a_1a_2(f+2))]/2,\\ c_{02}\ &=(f+1)a_{12}+3b,\ c_{10}=2a-a_{21},\ c_{01}=a_{12}-2b,\\ d\ &=(3a_{21}-2a+2f+3)/2,\ g=(3a_{30}-3a_{12}+2b+2c)/2 \end{aligned}$$

and a_{30}, a_{21}, a_{12} are the solutions of the following systems of algebraic equations:

$$\begin{aligned}
F_{50} &\equiv a_{30}(a_{12}^2+2a_{21})((f+2)(a_1+a_2)-c)+(a_{21}a_1a_2-a_{12}a_{30})(a-1)-\\
&-a_{30}(a_{12}^3+3a_{12}a_{21}+3a_{30})(f+1)+3a_{30}(a_1+a_2)(a_1a_2+a-1)+\\
&+a_{12}a_{30}((a_1+a_2)(c-a_1-a_2)-a_1a_2(f+1))-3ca_{30}a_1a_2=0,\\
F_{41} &\equiv (a_{12}^2a_{21}+a_{12}a_{30}+2a_{21}^2)((f+2)(a_1+a_2)-c)-\\
&-(f+1)(a_{12}^3a_{21}+a_{12}^2a_{30}+3a_{12}a_{21}^2+5a_{21}a_{30})+\\
&+2a_{21}(a_1+a_2)(a_1a_2-1+a)-2a_1a_2(ca_{21}-a_{12}(a-1))-\\
&-(a_{12}a_{21}+3a_{30})(a-1+a_1a_2(f+1)+(a_1+a_2)(a_1+a_2-c))=0,\\
F_{32} &\equiv (a_{12}^3+3a_{12}a_{21}+3a_{30})((f+2)(a_1+a_2)-c)+\\
&+a_{12}(a_1a_2+a-1)(a_1+a_2)+a_1a_2(3-3a-ca_{12})-\\
&-(a_{12}^4+4a_{12}^2a_{21}+4a_{12}a_{30}+2a_{21}^2)(f+1)-\\
&-(a_{12}^2+2a_{21})(a-1+a_1a_2(f+1)+(a_1+a_2)(a_1+a_2-c))=0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
F_{40} &\equiv 2a_{12}^2((f+2)(a_1+a_2)-(f+1)a_{12}-c)-3a_{12}a_{21}(2f+1)+\\
&+a_{12}(5-2a+2c(a_1+a_2)-2(a_1+a_2)^2-2(f+1)a_1a_2)+\\
&+2a_{21}(2(f+2)(a_1+a_2)-b-3c)+a_{30}(2a-6f-9)+\\
&+4(a_1+a_2)(a-1)-a_{21}a_{30}+6a_1a_2(a_1+a_2-c)-2b-2c=0,\\
F_{31} &\equiv (4b+6c)a_{30}-(2f-4)a_{12}^2-8a_{12}a_{30}+2a_{21}(a-3f-5)+\\
&+2a_{12}((a_1+a_2)(f+2)-2b-3c)+a_1a_2(4a-6f-20)-\\
&-a_{21}^2-4(a_1^2+a_2^2)+4c(a_1+a_2)-2a-2f-1=0,\\
F_{22} &\equiv a_{12}^2((f+1)a_{12}-(f+2)(a_1+a_2)+c)+3a_{12}a_{21}(f+2)+\\
&+a_{12}((a_1+a_2)^2+(f+1)a_1a_2-c(a_1+a_2)+3f+6)-\\
&-(a_{21}+1)(b+2(f+2)(a_1+a_2))-a_1a_2(a_1+a_2-c)=0,\\
F_{13} &\equiv (a_{12}-a_1)(a_{12}-a_2)(f+2)=0.
\end{aligned} \tag{9}$$

Denote $j_1=a_{12}(a_1+a_2)-3a_1a_2-a_{12}^2-2a_{21},\ j_2=f+1,\ j_3=a_2^3-a_2^2a_{12}-a_2a_{21}-a_{30},\ j_4=a_1^3-a_1^2a_{12}-a_1a_{21}-a_{30},\ j_5=4a_{12}^3a_{30}-a_{12}^2a_{21}^2+18a_{12}a_{21}a_{30}-4a_{21}^3+27a_{30}^2$.

In order to find the conditions for the existence of an invariant cubic for system (5) we study the consistency of the system of equations {(8), (9)} assuming that

$$(a_1-a_2)(f+2)\neq 0. \tag{10}$$

The case $f=-2$ was investigated in [12]. Then from the equation $F_{13}=0$ of (9), we can see that either $a_{12}=a_1$ or $a_{12}=a_2$.

3.1 Case $\mathbf{a_{12}=a_1,\ a_{21}=-1}$

Suppose that $a_{12}=a_1$ and $a_{21}=-1$. Then $F_{13}\equiv 0$ and $F_{22}\equiv 0$. We divide the investigation into two cases: $a_2=1/a_1$ and $a_2\neq 1/a_1$.

3.1.1. $a_2=1/a_1$. In this case $e_1\equiv F_{31}+2F_{32}=0$ implies

$$e_1\equiv[(f+2)a_1^2-2ba_1-3f-6](a_{30}-a_1)=0.$$

If $a_{30}=a_1$, then the cubic curve (6) is reducible. If $a_{30}-a_1\neq 0$, then the equation $e_1=0$ yields $b=[(a_1^2-3)(f+2)]/(2a_1)$.

We calculate the resultant of F_{50} and F_{40} with respect to f. We find that $Res(F_{50}, F_{40}, f) = (a_{30} - a_1)(a-1)f_1 f_2$, where $f_1 = a_1 a_{30} - 1$, $f_2 = a_1^2 - 3a_1 a_{30} - 2$.

Let $a = 1$. Since $F_{40} = 0$, we have $(f+1)f_2 = 0$. Suppose that $f_2 = 0$, then $a_{30} = (a_1^2 - 2)/(3a_1)$. In this case $F_{50} \equiv 0$, $F_{40} \equiv 0$ and $F_{31} \equiv (a_1^2 - 3)(f+1) = 0$.

If $a_1^2 = 3$, then we obtain the following set of conditions

1) $a = 1,\ b = 0,\ d = f - 1,\ g = (3c - 4a_1)/3,\ a_1^2 = 3,\ a_2 = a_1/3.$

for the existence of an invariant cubic $12(x^2 + y^2)(1 - y) + (c - g)(x^3 + 9xy^2) = 0$.

The case $a_1^2 \neq 3$ and $f = -1$ implies $c = (a_1^2 + 1)/a_1$ and is contained in 8).

Suppose that $f_2 \neq 0$ and $f = -1$. Then $F_{32} = 0$ implies $c = (a_1^2 + 1)/a_1$ and this case is contained in 8).

Let $a \neq 1$ and $f_1 = 0$. Then $a_{30} = 1/a_1$. We express a from $F_{40} = 0$, c from $F_{32} = 0$ and obtain that $F_{41} \neq 0$.

Let $(a-1)f_1 \neq 0$ and $f_2 = 0$. Then $a_{30} = (a_1^2 - 2)/(3a_1)$ and $F_{40} \neq 0$.

3.1.2. $a_2 \neq 1/a_1$. In this case we express c from $F_{40} = 0$ and the equation $e_2 \equiv F_{32} + F_{31} = 0$ implies

$$e_2 \equiv ((a_1 - 3a_2)(f+2) - 2b)(a_1 - a_{30}) = 0.$$

When $a_{30} = a_1$ the cubic curve is reducible. Assume that $a_{30} - a_1 \neq 0$. Then $e_2 = 0$ yields $b = ((a_1 - 3a_2)(f+2))/2$ and $F_{50} \equiv (a-1)(a_2 - a_{30}) = 0$.

Suppose that $a_{30} = a_2 = 0$. Then $F_{32} \equiv (a_1 - 2)(a_1 + 2)(f - a + 2) = 0$ and we obtain the following three sets of conditions for the existence of an invariant cubic:

2) $b = f + 2,\ c = 2 - a - f,\ d = f - a,\ g = 1 - a,\ a_1 = 2,\ a_2 = 0.$

The invariant cubic is $x^2 + y^2 - y(x - y)^2 = 0$.

3) $b = -f - 2,\ c = a + f - 2,\ d = f - a,\ g = a - 1,\ a_1 = -2,\ a_2 = 0.$

The invariant cubic is $x^2 + y^2 - y(x + y)^2 = 0$.

4) $c = [2b(2 - a)]/a,\ d = -2,\ f = a - 2,\ g = [b(1 - a)]/a,\ a_1 = (2b)/a,\ a_2 = 0.$

The invariant cubic is $a(x^2 + y^2)(y - 1) - 2bxy^2 = 0$.

Suppose that $a_{30} = a_2$ and $a_2 \neq 0$. We calculate the resultant of F_{41} and F_{32} with respect to f. We find that $Res(F_{41}, F_{32}, f) = (a_2 - a_1)^2(a-1)j_5$, where

$$j_5 = 4a_1^3 a_2 - a_1^2 - 18a_1 a_2 + 27a_2^2 + 4.$$

If $a = 1$ and $f = -1$, then this case is contained in 6). If $a = 1$ and $f \neq -1$, then we get the following set of conditions for the existence of an invariant cubic:

5) $a = 1,\ d = f - 1,\ b = [(f+2)(a_1 - 3a_2)]/2,\ g = [3(a_2 - a_1) + 2(b + c)]/2,$ $c = [2a_1 a_2(a_1 + a_2) + (f-1)a_1 - a_2(5f + 7)]/[2(a_1 a_2 - 1)],\ F_{41} \equiv 4a_1^2 a_2^2 - a_1^2 - 6a_1 a_2 + 15a_2^2 + 4 = 0,\ F_{32} \equiv 2a_1^3 a_2 - 4a_1^2 a_2^2 - a_1^2 + 6a_1 a_2^3 - 10a_1 a_2 + 19a_2^2 + 4 = 0.$

The invariant cubic is $(x^2 + y^2)(1 - y) + x(a_1 y^2 + a_2 x^2) = 0$.

Let $a \neq 1$ and $j_5 = 0$. The equation $j_5 = 0$ admits the following parametrization

$$a_2 = u^2(a_1 + 2u),\ a_1 = (-3u^2 - 1)/(2u).$$

In this case $F_{32} \equiv F_{41} \equiv ((f+1)u^2 + 2a - f - 3)(3u^2 - 1) = 0$. We obtain the following two sets of conditions for the existence of an invariant cubic:

6) $a = [f + 3 - (f + 1)u^2]/2,\ d = f - a,\ b = -[(f + 2)(3u^4 + 1)]/(4u),\ c = [f(u^2 + 1) + u^4 - 3u^2]/(2u),\ g = [((3f + 1)u^2 + f + 1)(1 - u^2)]/(4u),\ a_1 = (-3u^2 - 1)/(2u),\ a_2 = u^2(a_1 + 2u).$

The invariant cubic is $2u(x^2 + y^2) + (u^2x - 2uy - x)(ux + y)^2 = 0.$

7) $b = (-f - 2)/(3u),\ c = (9a + 3f - 16)/(9u),\ d = f - a,\ g = (9a - 10)/(9u),\ 3u^2 - 1 = 0,\ a_1 = (-1)/u,\ a_2 = (-u)/3.$

The invariant cubic is $9u(x^2 + y^2)(y - 1) + x(x^2 + 9y^2) = 0.$

Suppose that $a_2(a_{30} - a_2) \neq 0$ and $a = 1$. Then $F_{50} \equiv 0$. If $f = -1$, then $F_{41} \equiv F_{32} \equiv 0$ and we obtain the following set of conditions

8) $a = 1,\ d = -2,\ f = -1,\ a_1 = (2b + 3c)/4,\ a_2 = (c - 2b)/4$

for the existence of an invariant cubic curve

$$12(x^2 + y^2)(1 - y) + 3(2b + 3c)xy^2 + (c - 2b + 8g)x^3 = 0.$$

Assume that $f \neq -1$. Then we calculate the resultant of F_{41} and F_{32} with respect to a_2. We find that $Res(F_{41}, F_{32}, a_2) = -2(a_1 - a_{30})^2h_1h_2^2$, where

$$h_1 = a_1^2 - 3a_1a_{30} - 2,\quad h_2 = 4a_1^3a_{30} - a_1^2 - 18a_1a_{30} + 27a_{30}^2 + 4.$$

Let $h_1 = 0$. Then $F_{32} = 0$ yields $a_1^2 = 3$ and we get the following set of conditions

9) $a = 1,\ b = [(f + 2)(a_1 - 3a_2)]/2,\ c = (a_1a_2 - f + 2)/a_1,\ d = f - 1,\ g = [(b + c)a_1 - 4]/a_1,\ a_1^2 = 3$

for the existence of an invariant cubic $3a_1(x^2 + y^2)(1 - y) + x^3 + 9xy^2 = 0.$

Let $h_1 \neq 0$ and $h_2 = 0$. The equation $h_2 = 0$ admits the following parametrization $a_1 = -(3u^2 + 1)/(2u)$, $a_{30} = (u^3 - u)/2$. In this case we have $F_{32} \equiv (2ua_2 - u^2 + 1)(a_2 + u) = 0$ and obtain the following two sets of conditions for the existence of an invariant cubic:

10) $a = 1,\ b = [(f + 2)(3u^2 - 1)]/(4u),\ c = (f - fu^2 - 6u^2)/(2u),\ d = f - 1,\ g = (3u^4 + fu^2 + f + 1)/(4u),\ a_1 = (-3u^2 - 1)/(2u),\ a_2 = -u.$

The invariant cubic is $2u(x^2 + y^2) + (u^2x - 2uy - x)(ux + y)^2 = 0.$

11) $a = 1,\ b = [(f + 2)(1 - 3u^2)]/(2u),\ c = (fu^2 - 1)/u,\ d = f - 1,\ g = [(3u^2 - 2f - 3)(u^2 - 1)]/(4u),\ a_1 = (-3u^2 - 1)/(2u),\ a_2 = (u^2 - 1)/(2u).$

The invariant cubic is $2u(x^2 + y^2) + (u^2x - 2uy - x)(ux + y)^2 = 0.$

3.2 Case $\mathbf{a_{12} = a_1,\ a_{21} \neq -1}$

Let $a_{12} = a_1$ and $a_{21} \neq -1$. Then $F_{22} \equiv 0$ yields $b = (f + 2)(a_1 - 2a_2)$.

3.2.1. $j_1 = 0$, $a_{30} = (a_1a_2(a_1 - 2a_2))/3$. In this case $a_{21} = -a_1a_2$ and $F_{50} \equiv (a - 1)a_1a_2f_1f_2 = 0$, $F_{41} \equiv (a_1 + a_2 - c)a_1a_2f_1f_2 = 0$, $F_{32} \equiv (f + 1)a_1a_2f_1f_2 = 0$, where $f_1 = a_1 + a_2$, $f_2 = a_1 - 3a_2$.

Suppose that $a_1 = 0$. Then the equation $F_{40} = 0$ implies $c = 2(f + 1 + a)a_2$ and

$$F_{31} \equiv (2a + 2f + 1)(2a_2 + 1)(2a_2 - 1) = 0.$$

In this case we obtain the following three sets of conditions for the existence of an invariant cubic $x^2 + y^2 - y^3 = 0$ for system (5):

12) $b = -f - 2,\ c = f + 1 + a,\ d = (2f + 3 - 2a)/2,\ g = a - 1,\ a_1 = 0,\ a_2 = 1/2;$

13) $b = f + 2,\ c = -f - a - 1,\ d = (2f + 3 - 2a)/2,\ g = 1 - a,\ a_1 = 0,\ a_2 = -1/2;$

14) $a = (-2f - 1)/2,\ b = 2c(-f - 2),\ d = 2(f + 1),\ g = b + c,\ a_1 = 0,\ a_2 = c.$

Suppose that $a_1 \neq 0$ and $a_2 = 0$. Then $F_{40} = 0$ implies $c = a_1(2a - 2f - 3)/2$ and $F_{31} = 0$ yields $a = (-2f - 1)/2$. We have the following set of conditions

15) $a = (-2f-1)/2,\ d = 2(f+1),\ c = -2b(f+1)/(f+2),\ g = -b(2f+3)/(2f+4),$ $a_1 = b/(f + 2),\ a_2 = 0$

for the existence of an invariant cubic $bxy^2 + (f + 2)(x^2 + y^2 - y^3) = 0$.

Suppose that $a_1a_2 \neq 0$ and $a_2 = -a_1$. Since $F_{40} = 0$, we obtain $c = a_1(a_1^2 - 6f - 2a - 7)/2$ and $F_{31} \equiv (2a + 2f + 1 - a_1^2)(3a_1^2 - 1)(a_1^2 + 1) = 0$.

If $a_1^2 = 1/3$, then we get the following set of conditions for the existence of an invariant cubic:

16) $b = 3(f + 2)a_1,\ c = [-a_1(3a + 9f + 10)]/3,\ d = f - a + 2,\ g = [a_1(2 - 3a)]/3,$ $a_2 = -a_1,\ a_1^2 = 1/3.$

The invariant cubic is $3(x^2 + y^2) - (a_1x - y)(x^2 - 3y^2) = 0$.

If $a_1^2 \neq 1/3$ and $2a + 2f + 1 - a_1^2 = 0$, then we get the following set of conditions

17) $a = (a_1^2 - 2f - 1)/2,\ b = 3(f + 2)a_1,\ c = (-2f - 3)a_1,\ d = 2a + 4f + 3,$ $g = (-3a - 2f)a_1,\ a_2 = -a_1.$

The invariant cubic is $x^2 + y^2 - (a_1x + y)(a_1x - y)^2 = 0$.

Suppose that $a_1a_2(a_2 + a_1) \neq 0$ and $a_1 = 3a_2$. Then express c from $F_{40} = 0$ and $F_{31} = 0$ yields $2a + 2f + 1 + 3a_2^2 = 0$. In this case we obtain the following set of conditions

18) $b = (f + 2)a_2,\ d = 2a + 4f + 3,\ g = c - a_2(a + 3),\ c = a_2(1 - 2af + 17a - 2f^2 + 7f)/[3(a + f + 1)],\ 3a_2^2 + 2a + 2f + 1 = 0,\ a_1 = 3a_2.$

The invariant cubic is $x^2 + y^2 + (a_2x - y)^3 = 0$.

Suppose that $a_1a_2(a_2 + a_1)(a_1 - 3a_2) \neq 0$. Then the equations of (8) implies $a = 1,\ f = -1,\ c = a_1 + a_2$ and the system of equations (9) has no real solutions.

3.2.2. $j_1 = 0,\ a_{30} \neq (a_1a_2(a_1 - 2a_2))/3$. In this case we express c from $F_{32} = 0$ and a from $F_{41} = 0$, then $F_{50} \equiv j_2j_3j_4j_5 = 0$.

Assume that $j_2 = 0$. Then $f = -1$ and $F_{40} = 0$ yields $a_{30} = -b$. In this case the system of equations (9) has no real solutions.

Suppose that $j_2 \neq 0$ and $j_3 = 0$. In this case $a_{30} = a_2^3$ and the system (9) is consistent iff $(a_1 - a_2)(f + 2) = 0$, in contradictions with assumption (10).

Let $j_2 j_3 \neq 0$ and $j_4 = 0$, then $a_{30} = a_1^2 a_2$. This case is contained in 27).

Now let $j_2 j_3 j_4 \neq 0$ and $j_5 = 0$. Then $j_5 = 0$ admits the following parametrization
$$a_2 = v(36 - 5v)/(12u(v - 9)),\ a_1 = (v - 9)/u,\ a_{30} = v^2(4v - 27)/(108u^3).$$
We get the following set of conditions for the existence of an invariant cubic:

19) $a = (18u^2 - v(f + 1)(4v - 27))/(18u^2),\ c = (2fv - 18f + v)(54 - 7v)/[12(v - 9)u],\ b = [(f + 2)(11v^2 - 144v + 486)]/[6(v - 9)u],\ g = [(4v - 27)v^2 + 72u^3(c + b) - 108(v - 9)u^2]/(72u^3),\ d = [4(2f + 3 - 2a)u^2 + v(5v - 36)]/(8u^2),\ f = [3888(v - 9)u^4 + 72u^2(128v^3 - 2889v^2 + 21384v - 52488) - v^3(167v^2 - 2277v + 7776)]/[2592u^4(9 - v) - 72u^2(62v^3 - 1431v^2 + 10692v - 26244) + 8v^2(4v - 27)^2(2v - 9)],\ F_{40} \equiv 1296(v - 9)^2u^4 - 36vu^2(11v^2 - 144v + 486)(5v - 36) - v^3(47v^2 - 603v + 1944)(v - 9) = 0,\ a_2 = v(36 - 5v)/(12u(v - 9)),\ a_1 = (v - 9)/u.$

The invariant cubic is $108u^3(x^2 + y^2) + (4vx - 27x - 3uy)(vx + 6uy)^2 = 0$.

3.2.3. $j_1 \neq 0, j_2 = 0$. In this case $f = -1$, we express a from $F_{32} = 0$ and obtain
$$\begin{aligned} F_{50} \equiv\ & [4a_{21}^2 a_{30} + a_{21}^2 a_1 a_2(a_1 - 2a_2) + 2a_{21}a_{30}(a_1^2 + 5a_1a_2 - 3a_2^2) + \\ & + 3a_{30}^2(2a_1 + 3a_2) + 4a_{30}a_1^2a_2^2](c - a_1 - a_2) = 0, \\ F_{41} \equiv\ & [4a_{21}^3 + a_{21}^2(a_1^2 + 6a_1a_2 - 4a_2^2) + 2a_{21}a_{30}(3a_2 - 2a_1) + \\ & + 2a_{21}a_1^2a_2(a_1 - a_2) - 9a_{30}^2 + 2a_{30}a_1a_2(a_1 - 3a_2)](c - a_1 - a_2) = 0. \end{aligned}$$

Assume that $c = a_1 + a_2$, then $F_{50} \equiv 0$ and $F_{41} \equiv 0$. In this case the system of equations (9) has no real solutions.

Assume that $a_1 + a_2 - c \neq 0$. Then we calculate the resultant of F_{50} and F_{41} with respect to a_2. We find that $Res(F_{50}, F_{41}, a_2) = f_1 f_2 j_5$, where $f_1 = a_1a_{21} + a_{30}$, $f_2 = a_1^2a_{21} + 3a_1a_{30} + 2a_{21}^2,\ j_5 = 27a_{30}^2 + 2a_1a_{30}(2a_1^2 + 9a_{21}) - a_{21}^2(a_1^2 + 4a_{21})$.

Let $f_1 = 0$, then $a_{30} = -a_1a_{21}$ and $F_{41} \equiv (a_1^2 - a_{21})(a_2^2 - a_{21})a_{21} = 0$.

The cases $a_{21} = 0$ and $a_{21} = a_1^2$ are contained in 25) and 26), respectively. Suppose that $a_{21} = a_2^2$. We express a from $F_{31} = 0$ and a_1 from $F_{40} = 0$. In this case we have the following set of conditions

20) $a = (a_2^3 - a_2 + 2c)/(2a_2),\ d = (a_2^3 + a_2 - c)/a_2,\ f = -1,\ b = a_1 - 2a_2,\ g = (2c - 3a_1a_2^2 - a_1 - 4a_2)/2,\ a_1 = (a_2 - 5a_2^3 + 3ca_2^2 - c)/(2a_2^2).$

for the existence of an invariant cubic $x^2 + y^2 - (a_1x - y)(a_2x - y)(a_2x + y) = 0$.

Let $f_1 \neq 0$ and $f_2 = 0$. Then $a_{30} = -a_{21}(a_1^2 + 2a_{21})/(3a_1)$ and $F_{41} = 0$ yields $a_{21} = -a_1^2/3$. We express c from $F_{31} = 0$ and this case is contained in 29).

Let $f_1 f_2 \neq 0$ and $j_5 = 0$. The equation $j_5 = 0$ admits the following parametrization $a_{30} = [(4a_1v + 9)(a_1v + 9)^2]/(108v^3),\ a_{21} = [(5a_1v + 9)(a_1v + 9)]/(12v^2)$. In this case $F_{50} \equiv h_1h_2 = 0$, where $h_1 = a_1v + 6a_2v + 9,\ h_2 = 4a_1v - 3a_2v + 9$.

If $h_1 = 0$, then $F_{31} = 0$ has no real solutions. If $h_1 \neq 0,\ h_2 = 0$, then $v = 9/(3a_2 - 4a_1)$ and express c from $F_{31} = 0$. In this case $F_{40} \equiv e_1e_2 = 0$, where $e_1 = 3a_1^2 - 2a_1a_2 - a_2^2 + 4,\ e_2 = a_1^2 - 6a_1a_2 + 5a_2^2 - 4$.

Suppose that $e_1 = 0$. The equation $e_1 = 0$ admits the following parametrization $a_1 = (u^2 - 4)/(4u)$, $a_2 = (-3u^2 - 4)/(4u)$ and this case is contained in 24).

Suppose $e_1 \neq 0$ and $e_2 = 0$. The equation $e_2 = 0$ have the following parametrization $a_1 = (5u^2 - 4)/(4u)$, $a_2 = (u^2 - 4)/(4u)$. This case is contained in 23).

3.2.4. $j_1 j_2 \neq 0$, $j_3 = 0$. In this case $a_{30} = a_2(a_2^2 - a_1 a_2 - a_{21})$. We express a from $F_{32} = 0$ and obtain that $F_{41} \equiv g_1 g_2 g_3 g_4 = 0$, where $g_1 = a_2^2 - a_{21}$, $g_2 = 2a_1 a_2 - 3a_2^2 + a_{21}$, $g_3 = a_1^2 + 2a_1 a_2 - 3a_2^2 + 4a_{21}$, $g_4 = (a_1 - a_2)f - 2a_2 + c$.

Assume that $g_1 = 0$. If $f = (-3)/2$, then this case is contained in 27). If $2f + 3 \neq 0$, then we obtain the following set of conditions for the existence of an invariant cubic:

21) $a = (a_2^2 + 2f + 5)/2$, $b = -(f + 2)(3a_2^2 + 1)/(2a_2)$, $c = (fa_2^2 + 3a_2^2 + f + 1)/a_2$, $d = a_2^2 - 1$, $g = (2f + 3 - 3a_2^4 - 2fa_2^2)/(4a_2)$, $a_1 = (a_2^2 - 1)/(2a_2)$.

The invariant cubic is $2a_2(x^2 + y^2) - (a_2^2 x - 2a_2 y - x)(a_2 x + y)(a_2 x - y) = 0$.

Assume that $g_1 \neq 0$ and let $g_2 = 0$. Then $a_{21} = a_2(3a_2 - 2a_1)$ and $F_{41} \equiv 0$. In this case we get the following set of conditions

22) $a = (2 - u^2(a_2^2 + 2a_2 u - 3))/[2(u^2 + 1)]$, $f = (-a_2^2 - 2a_2 u - 4u^2 - 3)/[2(u^2 + 1)]$, $g = (3a_1 a_2^2 - 3a_1 - 6a_2^3 + 2b + 2c)/2$, $d = (3 + 2f - 2a - 6a_1 a_2 + 9a_2^2)/2$, $2u^2(c + 11b - 4u) + u((2b + c)^2 - 9) + 18b = 0$, $a_1 = 2a_2 + u$, $a_2 = (c - 2u + 2b)/3$.

for the existence of an invariant cubic $x^2 + y^2 + (ux - y)(a_2 x - y)^2 = 0$.

Assume that $g_1 g_2 \neq 0$ and let $g_3 = 0$. Then $a_{21} = (3a_2^2 - a_1^2 - 2a_1 a_2)/4$ and $F_{41} \equiv 0$. In this case we express c from $F_{31} = 0$ and find that $F_{40} \equiv s_1 s_2 = 0$, where $s_1 = a_1^2 - 6a_1 a_2 + 5a_2^2 - 4$, $s_2 = (2f + 5)a_1^2 - (4f + 6)a_1 a_2 + (2f + 1)a_2^2 + 8f + 12$.

If $s_1 = 0$, then this equation admits the following parametrization $a_1 = (5u^2 - 4)/(4u)$, $a_2 = (u^2 - 4)/(4u)$. In this case we obtain the following set of conditions for the existence of an invariant cubic:

23) $a = (8fu^2 + (u^2 + 4)^2)/[4(4 - u^2)]$, $d = (8f - 8a + 12 - 3a_1^2 - 6a_1 a_2 + 9a_2^2)/8$, $b = ((f + 2)(3u^2 + 4))/(4u)$, $c = ((3 - f)u^4 + 12fu^2 + 16)/[2u(u^2 - 4)]$, $g = (3a_2(a_1 - a_2)^2 - 12a_1 + 8(b + c))/8$, $a_1 = (5u^2 - 4)/(4u)$, $a_2 = (u^2 - 4)/(4u)$.

The invariant cubic is $16u(x^2 + y^2) + (u^2 x - 4x - 4uy)(ux - 2y)^2 = 0$.

Let $s_1 \neq 0$ and $s_2 = 0$. In this case we get the following set of conditions for the existence of an invariant cubic:

24) $b = [((8a + 5u^2 - 12)u^2 + 32(a - 1))(a - 1)]/u^5$, $c = [(u^2 + 8 - 8a)(u^2 + 2)]/u^3$, $d = [(2a - 5)u^2 + 16(a - 1)]/u^2$, $f = 2(2a - 2 - u^2)/u^2$, $g = [a_2 u(16a_2 + 3u^3 + 20u)]/[8(u^2 + 4)]$, $a_1 = a_2 + u$, $a_2 = [32(1 - a) + u^2(12 - 8a - u^2)]/(4u^3)$.

The invariant cubic is $4(x^2 + y^2) + (a_1 x - a_2 x - 2y)^2(a_2 x - y) = 0$.

Assume that $g_1 g_2 g_3 \neq 0$ and let $g_4 = 0$. Then $c = (f + 2)a_2 - fa_1$ and $F_{41} \equiv 0$. In this case the system of equations $\{F_{40} = 0, F_{31} = 0\}$ is consistent if and only if $(a_1 - a_2)(f + 2) = 0$, in contradiction with assumption (10).

3.2.5. $j_1 j_2 j_3 \neq 0,\ j_4 = 0$. In this case $a_{30} = -a_1 a_{21}$. We express a from $F_{32} = 0$ and obtain that $F_{41} \equiv h_1 h_2 h_3 = 0$, where $h_1 = a_{21},\ h_2 = a_1^2 - a_{21},\ h_3 = a_1 + a_2 - c$.

Let $h_1 = 0$. If $f = (-3)/2$, then this case is contained in 27). If $2f + 3 \neq 0$, then we get the following set of conditions for the existence of an invariant cubic:

25) $a = 1,\ b = -[(f+2)(4a_2^2+1)]/(4a_2),\ c = (4a_2^2 + f + 1)/(2a_2),\ d = (2f+1)/2,$ $g = [(2f+3)(1-4a_2^2)]/(8a_2),\ a_1 = (4a_2^2 - 1)/(4a_2).$

The invariant cubic is $4a_2(x^2+y^2) + (4a_2^2 x - 4a_2 y - x)y^2 = 0$.

Suppose that $h_1 \neq 0$ and $h_2 = 0$. Then $a_{21} = a_1^2$. We express c from $F_{31} = 0$ and $F_{40} \equiv i_1 i_2 = 0$, where $i_1 = 2f + 3,\ i_2 = (5a_1^2 - 4a_1 a_2 + 2)a_1^2 + 4a_2(a_1 - a_2) + 1$.

Let $i_1 = 0$, then $f = (-3)/2$ and this case is contained in 27). Assume that $i_1 \neq 0$ and let $i_2 = 0$. The equation $i_2 = 0$ admits the following parametrization $a_1 = (u^2-1)/(2u),\ a_2 = (5u^4 - 2u^2 + 1)/(8u^3)$. In this case we find the following set of conditions for the existence of an invariant cubic:

26) $a = (4fu^2 - 4f + u^4 + 12u^2 - 5)/(8u^2),\ c = (4fu^4 + 4fu^2 + 15u^4 + 1)/(8u^3),$ $b = -[(f+2)(3u^4+1)]/(4u^3),\ d = (2fu^2 + 2f + u^4 - 3u^2 + 4)/(4u^2),\ g = -[(4f + 3u^2 + 3)(u^2-1)^2]/(16u^3),\ a_1 = (u^2-1)/(2u),\ a_2 = (5u^4 - 2u^2 + 1)/(8u^3).$

The invariant cubic is $x^2 + y^2 - (a_1 x + y)(a_1 x - y)^2 = 0$.

Suppose that $h_1 h_2 \neq 0$ and $h_3 = 0$. In this case $a_1 = c - a_2,\ f = (-3)/2$ and we get the following set of conditions for the existence of an invariant cubic:

27) $d = 2a - 3,\ f = (-3)/2,\ g = 2(1-a)(b+c),\ a_1 = 2(b+c)/3,\ a_2 = (c - 2b)/3.$

The invariant cubic is $3(x^2+y^2) - (2ax^2 - 2x^2 - y^2)(2bx + 2cx - 3y) = 0$.

3.2.6. $j_1 j_2 j_3 j_4 \neq 0,\ j_5 = 0$. The equation $j_5 = 0$ admits the following parametrization $a_{30} = (4a_1^3 u^3 + 81a_1^2 u^2 + 486a_1 u + 729)/(108u^3),\ a_{21} = (5a_1^2 u^2 + 54a_1 u + 81)/(12u^2)$. We reduce the equations $F_{50} = 0$ and $F_{41} = 0$ by a from $F_{32} = 0$. Then $F_{41} \equiv s_1 s_2 = 0$, where $s_1 = a_1 u + 3,\ s_2 = (7a_1 f + a_1 - 6a_2 + 6c)u + 9f + 9$.

Assume that $s_1 = 0$, then $a_1 = (-3)/u$. We express c from $F_{31} = 0$ and substituting in (9) we obtain that $F_{40} \equiv r_1 r_2 = 0$, where

$$r_1 = (2f+3)u^2 + 2f + 7,\ r_2 = (4a_2^2 - 1)u^4 + 12a_2 u^3 + 4a_2 u + 6u^2 + 3.$$

Let $r_1 = 0$, then $f = (-3u^2 - 7)/[2(u^2+1)]$ and we get the following set of conditions for the existence of an invariant cubic:

28) $a = (2u^4 + 3u^2 - 3)/[2u^2(u^2+1)],\ b = (2a_2 u + 3)(3 - u^2)/[2u(u^2+1)],\ c = [a_2 u(u^2+1) - 3u^2 - 7]/[u(u^2+1)],\ d = -(u^4 + 8u^2 + 3)/[u^2(u^2+1)],\ f = -(3u^2+7)/[2(u^2+1)],\ g = (8a_2 u^3 + u^2 - 3)/[2u^3(u^2+1))],\ a_1 = (-3)/u.$

The invariant cubic is $u^3(x^2+y^2) - (x + uy)^3 = 0$.

Assume that $r_1 \neq 0$ and let $r_2 = 0$. The equation $r_2 = 0$ admits the following parametrization $a_2 = (1 + 6v^2 - 3v^4)/(8v^3),\ u = (2v)/(v^2 - 1)$. In this case we have the following set of conditions for the existence of an invariant cubic:

29) $a = [4f(1-v^2)(3v^2-1)-(v^2+1)(3v^4+5)]/[8v^2(v^2-3)],\ b = -(f+2)(3v^4+1)/(4v^3),\ c = [(4f-17)v^6+(19-40f)v^4+(20f-31)v^2-3]/[8v^3(v^2-3)],\ d = [2(f+8)-3v^6+10(f+3)v^4-(20f+47)v^2]/[4v^2(v^2-3)],\ g = [(1-v^2)(3v^6+(4f+7)v^4+(48f+77)v^2+12f+9)]/[16v^3(v^2-3)],\ a_1 = 3(1-v^2)/(2v),\ a_2 = (1+6v^2-3v^4)/(8v^3)$.

The invariant cubic is $8v^3(x^2+y^2)-((v^2-1)x+2vy)^3=0$.

Suppose that $s_1 \neq 0$ and $s_2 = 0$. Then $a_2 = (7a_1fu+a_1u+6cu+9f+9)/(6u)$. We express c from $F_{31}=0$ and obtain that $F_{40} \equiv u^2(256f^2+768f+527)a_1^2+18u(64f^2+192f+137)a_1+9[16f(f+3)(u^2+9)+9(4u^2+35)]=0$. The equation $F_{40}=0$ has the following parametrization $a_1 = 3(3-3h^2+2hu)/[4u(h^2-1)]$, $f = (9-24h^2u-9h^2+14hu-24u)/[16u(h^2+1)]$. In this case we get the following set of conditions for the existence of an invariant cubic

30) $b = (f+2)(a_1-2a_2),\ f = (9-24h^2u-9h^2+14hu-24u)/[16u(h^2+1)],\ d = [u^2(12-8a+8f+5a_1^2)+54ua_1+81]/(8u^2),\ a = [4u^2(16h^6-12h^4-7h^3-12h^2+16)+36uh(2h^4-3h^3+3h-2)+81h(h^4-2h^2+1)]/[64u^2(h^2+1)(h^2-1)^2],\ g = [4u^3(a_1^3-27a_1+18b+18c)+81u^2a_1^2+486ua_1+729]/(72u^3),\ c = [4u^2(8h^6+34h^5-100h^4+117h^3-100h^2+34h+8)+36u(7-7h^6+18h^5-14h^4+14h^2-18h)+81h(h^4-2h^2+1)]/[64u^2(h^2+1)(h+1)(h-1)^3],\ a_1 = 3(3-3h^2+2hu)/[4u(h^2-1)],\ a_2 = (7a_1fu+a_1u+6cu+9f+9)/(6u)$.

The invariant cubic is $108u^3(x^2+y^2)+(4ua_1x-3uy+9x)(uxa_1+6uy+9x)^2=0$.

3.3 Case $\mathbf{a_{12} = a_2}$

The case $a_{12} = a_2$ is equivalent with $a_{12} = a_1$ if we take into consideration the symmetry $F_{ij}(a_1,a_2) = F_{ij}(a_2,a_1)$ in the algebraic system {(8), (9)}.

4 Sufficient conditions for the existence of a center

In this section we derive sixteen sets of sufficient conditions for the origin to be a center for cubic system (1) by constructing integrating factors or first integrals from invariant functions.

Lemma 1. *The following three sets of conditions are sufficient conditions for the origin to be a center for system (1):*

(i) $a=1,\ b=l=s=0,\ d=f-1,\ k=g=(ca_1-4)/a_1,\ m=(4ca_1+3f-13)/3,\ n=2r,\ p=(8-ca_1+4f)/a_1,\ q=-2g,\ r=-(f+1),\ a_1^2=3;$

(ii) $a=1,\ d=-2,\ f=-1,\ k=g,\ l=-b,\ m=(3c^2-4b^2-4bc-16)/16,\ n=-m,\ p=b,\ q=-g,\ r=s=0;$

(iii) $d = 2a - 3,\ f = (-3)/2,\ g = 2(1-a)(b+c),\ k = (1-a)(2b+c),\ l = -b,$ $m = (9a - 4b^2 - 2bc + 2c^2 - 9)/9,\ n = (18 - 18a + 2b^2 + bc - c^2)/9,\ p = (2b-c)/2,\ q = 2(a-1)(b+c),\ r = 1/2,\ s = 2(a-1)(2b-c)(b+c)/9.$

Proof. In Cases (i)–(iii), system (1) has a Darboux first integral of the form

$$l_2^{\alpha}\Phi = C.$$

In Case (i): $l_2 = x + a_1(1-y),\ \Phi = 12(x^2+y^2)(1-y) + (c-g)(x^3+9xy^2),\ \alpha = -3.$

In Case (ii): $l_2 = 4 + (c-2b)x - 4y,\ \Phi = 12(x^2+y^2)(1-y) + 3(2b+3c)xy^2 + (c - 2b + 8g)x^3,\ \alpha = -3.$

In Case (iii): $l_2 = 3(1-y) + (c-2b)x,\ \Phi = 3(x^2+y^2) - (2ax^2 - 2x^2 - y^2)(2bx + 2cx - 3y),\ \alpha = -2.$ □

Lemma 2. *The following seven sets of conditions are sufficient conditions for the origin to be a center for system (1):*

(i) $b = (-1)/5,\ a = -3b,\ c = 18b,\ d = 14b,\ f = 11b,\ g = -2b,\ k = q = 2b,$ $l = -b,\ m = n = -9b,\ p = 21b,\ r = -6b,\ s = 0;$

(ii) $b = 1/5,\ a = 3b,\ c = -18b,\ d = -14b,\ f = -11b,\ g = -2b,\ k = q = 2b,$ $l = -b,\ m = n = 9b,\ p = 21b,\ r = 6b,\ s = 0;$

(iii) $b = (-1)/5,\ d = 6b,\ f = 9b,\ p = b,\ r = -4b,\ l = n = -b,\ c = 1/10,$ $a = 9c,\ g = -c,\ m = -3c,\ q = c,\ k = (-3)/20,\ s = 0;$

(iv) $b = 1/5,\ d = -6b,\ f = -9b,\ l = -b,\ n = p = b,\ r = 4b,\ c = (-1)/10,$ $a = -9c,\ g = -c,\ m = 3c,\ q = c,\ k = 3/20,\ s = 0;$

(v) $a = (-2f-1)/2,\ c = -da_1,\ d = -2r,\ g = (na_1)/2,\ k = 2g,\ n = -2f-3,\ l = -b,\ m = n(2a_1^2 - 3)/2,\ p = -4g,\ q = -g,\ r = -f-1,\ s = 0,\ a_1 = b/(f+2);$

(vi) $a = (1 - 5f - 2f^2)/(2f+7),\ c = (1-2f)a_2,\ d = 2a + 4f + 3,\ g = c - (a+3)a_2,\ k = 4(a-1)a_2 + g,\ l = -b,\ m = [(11f+21)(2f+3)]/(2f+7),$ $n = [(2f+3)(f-4)]/(2f+7),\ p = (7f+9)a_2,\ q = 12a_2^3 - 3ca_2^2 - g,\ r = -f-1,\ s = 3(1-a)a_2^2,\ (2f+7)b^2 + (2f+3)(f+2)^2 = 0,\ a_2 = b/(f+2);$

(vii) $a = [2 - u^2(a_2^2 + 2a_2u - 3)]/[2(u^2+1)],\ f = (-a_2^2 - 2a_2u - 4u^2 - 3)/[2(u^2+1)],$ $g = (3a_1a_2^2 - 3a_1 - 6a_2^3 + 2b + 2c)/2,\ d = (3 + 2f - 2a - 6a_1a_2 + 9a_2^2)/2,$ $2u^2(c + 11b - 4u) + u((2b+c)^2 - 9) + 18b = 0,\ k = (a-1)(a_1+a_2) + g,$ $l = -b,\ s = (1-a)a_1a_2,\ m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1+a_2) - a + d + 2,$ $r = -f-1,\ n = a_1a_2(-f-2) - (d+1),\ p = (f+2)(a_1+a_2) + b - c,$ $q = (2b-u)a_1a_2 - g,\ a_2 = (c - 2u + 2b)/3,\ a_1 = (4b + 2c - u)/3.$

Proof. In Cases (i)–(vii), system (1) has a Darboux integrating factor of the form

$$\mu = l_1^{\alpha_1} l_2^{\alpha_2} \Phi^{\alpha_3}.$$

In Case (i): $l_1 = 1 + 2x - y,\ l_2 = 1 - y,\ \Phi = x^2 + y^2 - y(x-y)^2,\ \alpha_1 = 1,$ $\alpha_2 = 1/2,\ \alpha_3 = (-5)/2.$

In Case (ii): $l_1 = 1 - 2x - y$, $l_2 = 1 - y$, $\Phi = x^2 + y^2 - y(x + y)^2$, $\alpha_1 = 1$, $\alpha_2 = 1/2$, $\alpha_3 = (-5)/2$.

In Case (iii): $l_1 = 1 - y$, $l_2 = 2 + x - 2y$, $\Phi = x^2 + y^2 - y^3$, $\alpha_1 = 1/2$, $\alpha_2 = 1$, $\alpha_3 = (-5)/2$.

In Case (iv): $l_1 = 1 - y$, $l_2 = 2 - x - 2y$, $\Phi = x^2 + y^2 - y^3$, $\alpha_1 = 1/2$, $\alpha_2 = 1$, $\alpha_3 = (-5)/2$.

In Case (v): $l_1 = (f + 2)(1 - y) + bx$, $l_2 = 1 - y$, $\Phi = (f + 2)(x^2 + y^2) + (bx - y(f + 2))y^2$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = -2$.

In Case (vi): $l_1 = 3bx+(f+2)(1-y)$, $l_2 = bx+(f+2)(1-y)$, $\Phi = (f+2)^3(x^2 + y^2) + (bx - (f + 2)y)^3$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = -2$.

In Case (vii): $l_1 = (4b + 2c - u)x + 3(1 - y)$, $l_2 = (c + 2b - 2u)x + 3(1 - y)$, $\Phi = 9(x^2 + y^2) + ((2u - 2b - c)x + 3y)^2(xu - y)$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = -2$. □

Lemma 3. *The following six sets of conditions are sufficient conditions for the origin to be a center for system (1):*

(i) $a = 3/2$, $b = (7c)/6$, $d = -3$, $f = (-3)/2$, $g = (-11c)/6$, $k = -3p$, $l = -b$, $m = (-41)/6$, $n = 9/2$, $p = c/2$, $q = 7p$, $r = 1/2$, $s = 5/2$, $c^2 - 3 = 0$;

(ii) $a = 5/6$, $c = 6g$, $d = -3$, $f = (-13)/6$, $g = -5b$, $k = g/3$, $m = 19/54$, $l = -b$, $n = 37/18$, $p = (103b)/3$, $q = (25b)/3$, $r = 7/6$, $s = 1/18$, $108b^2 - 1 = 0$;

(iii) $a = 2/(5u^2)$, $b = (8 - 5u^2)/(20u)$, $c = (169u^2 - 76)/(10u^3)$, $d = (44 - 105u^2)/(20u^2)$, $f = (4 - 45u^2)/(20u^2)$, $g = (9u^2 - 4)/(2u^3)$, $k = (20 - 49u^2)/(10u^3)$, $l = -b$, $m = (1215u^2 - 508)/(25u^4)$, $n = (45u^2 - 24)/(10u^2)$, $p = 23(4 - 9u^2)/(10u^3)$, $q = 3(8 - 19u^2)/(5u^3)$, $r = -f - 1$, $s = (5u^2 - 2)/(5u^2)$, $5u^4 - 40u^2 + 16 = 0$;

(iv) $a = 7(11u^2 - 1)/(40u^4)$, $b = (7 - 85u^2)/(200u^5)$, $c = (185u^2 - 19)/(100u^5)$, $d = (5 - 47u^2)/(20u^2)$, $f = (1 - 75u^2)/(40u^2)$, $g = (1 - 3u^2)/(40u^5)$, $k = (9 - 35u^2)/(200u^5)$, $l = -b$, $m = (23 - 229u^2)/(200u^6)$, $n = (105u^2 - 11)/(200u^6)$, $p = (37 - 375u^2)/(200u^5)$, $q = (65u^2 - 11)/(200u^5)$, $r = -f - 1$, $s = (5u^2 + 1)/(200u^6)$, $5u^4 - 10u^2 + 1 = 0$;

(v) $a = (ha_1)/(h^2-1)$, $b = (ha_1)/(h-1)^2$, $c = [h(14h-11h^2-11)a_1]/[(h^2+1)(h-1)^2]$, $f = (h-2h^2-2)/(h^2+1)$, $d = [12(39h^3-49h^2+28h+7)]/[(h^2+1)(h^2-1)^2]$, $g = [24h(27h^3 - 35h^2 + 20h + 5)]/[(h^2 + 1)(1 - h^2)^3]$, $k = (a - 1)(a_1 + a_2) + g$, $l = -b$, $s = (1 - a)a_1a_2$, $m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1 + a_2) - a + d + 2$, $r = -f - 1$, $n = a_1a_2(-f - 2) - (d + 1)$, $p = (f + 2)(a_1 + a_2) + b - c$, $q = (a_1 + a_2 - c)a_1a_2 - g$, $a_2 = (-ha_1)/(h - 1)^2$, $a_1 = 2(h^2 - h + 1)/(1 - h^2)$, $h^4 + 4h^3 - 6h^2 + 4h + 1 = 0$;

(vi) $a = 2$, $c = [b(h - 2)(1 - 2h)]/(h^2 + 1)$, $f = (h - 2h^2 - 2)/(h^2 + 1)$, $b = (ha_1)/(h - 1)^2$, $d = (-6h^3)/[(h^2 + 1)(h^2 - 1)^2]$, $g = [6h(10h^3 + 3h^2 + 6h - 3)]/[(h^2 + 1)(h^2 - 1)^3]$, $k = (a - 1)(a_1 + a_2) + g$, $l = -b$, $s = (1 - a)a_1a_2$,

$m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1 + a_2) - a + d + 2,\ n = a_1a_2(-f - 2) - (d + 1),$ $r = -f - 1,\ p = (f + 2)(a_1 + a_2) + b - c,\ q = (a_1 + a_2 - c)a_1a_2 - g,\ a_2 =$ $(-ha_1)/(h - 1)^2,\ a_1 = 2(h^2 - h + 1)/(1 - h^2),\ h^4 - 2h^3 - 2h + 1 = 0.$

Proof. When conditions (i)–(vi) hold the cubic system (1) is Darboux integrable having three invariant straight lines and one invariant cubic curve. It has an integrating factor of the form $\mu = l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3} \Phi^{\alpha_4}$ in Cases (i)–(iv) and a first integral $l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3} \Phi^{\alpha_4} = C$ in Cases (v) and (vi).

In Case (i): $l_1 = c + 5x - cy,\ l_2 = 1 - cx - y,\ l_3 = c - 4x - 4cy,\ \Phi = 2c(x^2 + y^2) + (x - c^2x - 2cy)(y - cx)^2,\ \alpha_1 = \alpha_3 = (-3)/2,\ \alpha_2 = (-5)/2,\ \alpha_4 = 1.$

In Case (ii): $l_1 = 6b - x - 6by,\ l_2 = 3 - 6bx - 3y,\ l_3 = 27b - 2x - 36by,\ \Phi = 54b(x^2 + y^2)(y - 1) + x(x^2 + 9y^2),\ \alpha_1 = (-3)/2,\ \alpha_2 = \alpha_3 = (-5)/2,\ \alpha_4 = 1.$

In Case (iii): $l_1 = 4u + (5u^2 - 4)x - 4uy,\ l_2 = 4u + (u^2 - 4)x - 4uy,\ l_3 = 8u^2 + (7u^2 - 4)(ux - 2y),\ \Phi = 16u(x^2 + y^2) + (u^2x - 4x - 4uy)(ux - 2y)^2,\ \alpha_1 = \alpha_2 = \alpha_3 = 1,\ \alpha_4 = -3.$

In Case (iv): $l_1 = 2u + (u^2 - 1)x - 2uy,\ l_2 = u + x - uy,\ l_3 = 8u^3 + (1 - u^4)x - 2u(1 + u^2)y,\ \Phi = 8u^3(x^2 + y^2) - (u^2x - x + 2uy)(u^2x - x - 2uy)^2,\ \alpha_1 = \alpha_2 = \alpha_3 = 1,\ \alpha_4 = -3.$

In Case (v): $l_1 = (1 - h^2)(1 - y) + 2(h^2 - h + 1)x,\ l_2 = (h + 1)(h - 1)^3(1 - y) + 2h(h^2 - h + 1)x,\ l_3 = (h^4 - 1)(h - 1)^2 - (h^2 - h + 1)(h^2 - 4h + 1)(h^2x + x + h^2y - y),$ $\Phi = (h^2 - 1)^3(x^2 + y^2) + (2hx - h^2y + y)(h^2x + x + h^2y - y)^2,\ \alpha_1 = 1,\ \alpha_2 = \alpha_3 = 2,$ $\alpha_4 = -2.$

In Case (vi): $l_1 = (1 - h^2)(1 - y) + 2(h^2 - h + 1)x,\ l_2 = (1 - 2h)(y - 1) + (h^3 - h^2 + h)x,\ l_3 = 6h^3x + (h^2 + 1)(2h - 1)(1 - 3y),\ \Phi = (h^2 - 1)^3(x^2 + y^2) + (2hx - h^2y + y)(h^2x + x + h^2y - y)^2,\ \alpha_2 = 2,\ \alpha_1 = \alpha_3 = 1,\ \alpha_4 = -2.$ □

5 Solution of the problem of the center for a cubic system with a bundle of two invariant straight lines and one invariant cubic

Theorem 4. *Let the cubic system (1) have a bundle of two invariant straight lines (4) and one invariant cubic (6). Then a singular point $O(0,0)$ is a center if and only if the first three Lyapunov quantities vanish.*

Proof. To prove the theorem, we compute the first three Lyapunov quantities L_1, L_2 and L_3 in each series of conditions $1) - 30)$ obtained in Section 3 by using the algorithm described in [9]. In the expressions for L_j we will neglect the denominators and non-zero factors.

In Case 1) we have $L_1 = 0$, then Lemma 1, (i).

In Case 2) the first Lyapunov quantity is $L_1 = a^2 + f^2 - a + 3f + 2$. We calculate L_2 and reduce it by a^2 from $L_1 = 0$. Then $L_2 = 0$ yields $a = (8f^2 + 27f + 16)/(4f + 1)$ and L_1 becomes $L_1 = (8f^2 + 20f + 11)(5f + 11)(f + 1)$. If $f = -1$, then Lemma 1, (ii) $(b = 1,\ c = 2)$. If $f = (-11)/5$, then Lemma 2, (i).

Assume that $(f+1)(5f+11) \neq 0$ and let $8f^2+20f+11=0$. In this case the equation $L_1=0$ has real solutions and $L_3 \neq 0$. Therefore the origin is a focus.

In Case 3) the first Lyapunov quantity is $L_1 = a^2+f^2-a+3f+2$. We calculate L_2 and reduce it by a^2 from $L_1=0$. Then $L_2=0$ yields $a=(8f^2+27f+16)/(4f+1)$ and L_1 becomes $L_1=(8f^2+20f+11)(5f+11)(f+1)$. If $f=-1$, then Lemma 1, (ii) ($b=-1,\ c=-2$). If $f=(-11)/5$, then Lemma 2, (ii).

Suppose that $(f+1)(5f+11) \neq 0$ and $8f^2+20f+11=0$. In this case the equation $L_1=0$ has real solutions and $L_3 \neq 0$. Therefore the origin is a focus.

In Case 4) the vanishing of L_1 gives $a=1$, then Lemma 1, (ii) ($c=2b$).

In Case 5) the vanishing of the first Lyapunov quantity gives $a_1=3a_2$, then $F_{32} \equiv 9a_2^4-5a_2^2+1=0$ has no real solutions. In this case the origin is a focus.

In Case 6) the vanishing of the first Lyapunov quantity gives $f=2(u^2-2u^4-3)/(3u^4-2u^2+3)$, then $L_2=f_1f_2$, where $f_1=u^2-3,\ f_2=3u^6+5u^4+9u^2-9$. If $f_1=0$, then Lemma 3, (i). Assume $f_1 \neq 0$ and let $f_2=0$. The equation $f_2=0$ has real solutions and $L_3 \neq 0$. Therefore the origin is a focus.

In Case 7) the first Lyapunov quantity is $L_1=9a^2-13a+3f^2+9f+10$. We calculate L_2 and reduce it by f^2 from $L_1=0$. Then $L_2=0$ yields $f=(216a^2-519a+185)/[9(12a-5)]$ and L_1 becomes $L_1=(21a-10)(9a-4)(9a-10)(6a-5)$.

If $a=5/6$, then Lemma 3, (ii). Suppose that $6a-5 \neq 0$. If $a=10/9$ or $a=9/4$ or $a=10/21$, then $L_1=L_2=0$ and $L_3 \neq 0$. In these subcases the origin is a focus.

In Case 8) the first Lyapunov quantity vanishes, then Lemma 1, (ii).

In Cases 9), 10) and 11) we have $L_1 \neq 0$. Therefore the origin is a focus.

In Case 12) the first Lyapunov quantity is $L_1=2a^2-5a+2f^2+7f+9$. We reduce L_2 by a^2 from $L_1=0$. Then $L_2=0$ yields $a=(9-22f-16f^2)/[2(16f+27)]$ and L_1 becomes $L_1=(32f^2+120f+111)(5f+9)(2f+3)$. If $f=(-3)/2$, then Lemma 1, (iii) ($a=1,\ b=-1/2,\ c=1/2$). If $f=(-9)/5$, then Lemma 2, (iii).

Assume $(2f+3)(5f+9) \neq 0$ and let $32f^2+120f+111=0$. In this case the equation $L_1=0$ has real solutions and $L_3 \neq 0$. Therefore the origin is a focus.

In Case 13) the first Lyapunov quantity is $L_1=2a^2-5a+2f^2+7f+9$. We reduce L_2 by a^2 from $L_1=0$. Then $L_2=0$ yields $a=(9-22f-16f^2)/[2(16f+27)]$ and L_1 becomes $L_1=(32f^2+120f+111)(5f+9)(2f+3)$. If $f=(-3)/2$, then Lemma 1, (iii) ($a=1,\ b=1/2,\ c=-1/2$). If $f=(-9)/5$, then Lemma 2, (iv).

Assume that $(2f+3)(5f+9) \neq 0$ and $32f^2+120f+111=0$. In this case the equation $L_1=0$ has real solutions and $L_3 \neq 0$. Therefore the origin is a focus.

In Case 14) the vanishing of L_1 gives $f=(-3)/2$, then Lemma 1, (iii) ($a=1,\ c=-b$).

In Case 15) we find that $L_1=0$, then Lemma 2, (v).

In Case 16) the first Lyapunov quantity is $L_1=9a^2-21a+27f^2+99f+100$. We reduce L_2 by a^2 from $L_1=0$. Then $L_2=0$ yields $a=(18f^2+84f+89)/[6(3f+5)]$ and L_1 becomes $L_1=(72f^3+396f^2+720f+433)(2f+3)$. If $f=(-3)/2$, then

Lemma 1, (iii) ($c = 0,\ b^2 = 3/4$). Let $2f+3 \neq 0$ and $72f^3+396f^2+720f+433 = 0$. In this case the equation $L_1 = 0$ has real solutions and $L_3 \neq 0$. Therefore the origin is a focus.

In Case 17) the vanishing of L_1 gives $f = (-3)/2$, then Lemma 1, (iii) ($a = (2b^2+9)/9,\ c = 0$).

In Case 18) the vanishing of the first Lyapunov quantity gives $a = -(2f^2+5f-1)/(2f+7)$, then Lemma 2, (vi).

In Case 19) we calculate the resultant of F_{40} and L_1 with respect to v. We find that $Res(F_{40}, L_1, v) = (27556u^6 - 14040u^4 - 17496u^2 + 177147)(841u^4 - 4374u^2 + 6561)^2(17u^2+54u+81)^3(17u^2-54u+81)^3(u^2+1)^8u^{36} \neq 0$. The origin is a focus.

In Case 20) the first Lyapunov quantity is $L_1 = c^2(3a_2^4-1)+ca_2(2-9a_2^4+3a_2^2)+a_2^2(6a_2^4-7a_2^2-1)$. We reduce L_2 by c^2 from $L_1 = 0$. Then express c from $L_2 = 0$ and L_1 becomes $L_1 = 245025a_2^{22}+1239975a_2^{20}-429264a_2^{18}-5822568a_2^{16}+1182522a_2^{14}+5547390a_2^{12}-1322072a_2^{10}-1639888a_2^8+405789a_2^6+88067a_2^4-4688a_2^2+784$. The equation $L_1 = 0$ has real solutions and $L_3 \neq 0$. In this case the origin is a focus.

In Cases 21), 24), 25) and 28) we find that $L_1 \neq 0$. The origin is a focus.

In Case 22) the first Lyapunov quantity vanishes, then Lemma 2, (vii).

In Case 23) the vanishing of the first Lyapunov quantity gives $f = 4(6u^2 - 3u^4 - 8)/(5u^4 - 8u^2 + 16)$, then $L_2 = e_1e_2$, where $e_1 = 5u^4 - 40u^2 + 16,\ e_2 = 15u^6 - 12u^4 - 48u^2 - 64$. If $e_1 = 0$, then Lemma 3, (iii). Let $e_1 \neq 0$ and $e_2 = 0$. The equation $e_2 = 0$ has real solutions and $L_3 \neq 0$. Therefore the origin is a focus.

In Case 26) the vanishing of L_1 gives $f = (4u^2 - 17u^4 - 3)/[2(5u^4 - 2u^2 + 1)]$, then $L_2 = e_1e_2$, where $e_1 = 5u^4 - 10u^2 + 1,\ e_2 = 15u^6 - 3u^4 - 3u^2 - 1$. If $e_1 = 0$, then Lemma 3, (iv). Let $e_1 \neq 0$ and $e_2 = 0$. The equation $e_2 = 0$ has real solutions and $L_3 \neq 0$. Therefore the origin is a focus.

In Case 27) the first Lyapunov quantity vanishes, then Lemma 1, (iii).

In Case 29) the first Lyapunov quantity is $L_1 = Af^2 + Bf + C$, where $A = 4(9v^6 - 25v^4 + 27v^2 - 3)(v^2 + 1),\ B = 4(42v^8 - 89v^6 + 33v^4 + 85v^2 - 15),\ C = 195v^8 - 478v^6 + 284v^4 + 254v^2 - 63$.

Let $A = 0$. The equation $A = 0$ has real solutions and $L_1 = 0$ yields $f = (-C)/B$. In this case $L_2 \neq 0$.

Now let $A \neq 0$. We reduce L_2 by f^2 from $L_1 = 0$ and express f from $L_2 = 0$. Then L_1 becomes $L_1 = 405v^{16}-3456v^{14}+10260v^{12}-15328v^{10}+16054v^8-13248v^6+6340v^4-352v^2+93$. The equation $L_1 = 0$ has real solutions and $L_3 \neq 0$. In this case the origin is a focus.

In Case 30) the vanishing of L_1 gives $u = [9(h^2 - 1)]/[2(4h^2 - h + 4)]$, then $L_2 = e_1e_2$, where $e_1 = h^4 + 4h^3 - 6h^2 + 4h + 1,\ e_2 = h^4 - 2h^3 - 2h + 1$. If $e_1 = 0$, then Lemma 3, (v) and if $e_2 = 0$, then Lemma 3, (vi). □

References

[1] Amel'kin V. V., Lukashevich N. A., Sadovskii A. P. *Non-linear oscillations in the systems of second order*, Belarusian University Press, Belarus, 1982 (in Russian).

[2] Bautin N. N. *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Mat. Sb., 1952, **30**, No. 72, 181–196 (in Russian); Transl. Amer. Math. Soc., 1954, **100**, No. 1, 397–413.

[3] Bondar Y. L., Sadovskii A. P. *Variety of the center and limit cycles of a cubic system, which is reduced to Lienard form*. Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2004, No. 3 (46), 71–90.

[4] Chavarriga J., Giné J. *Integrability of cubic systems with degenerate infinity*, Differential Equations and Dynamical Systems, 1998, **6**, No. 4, 425–438.

[5] Cozma D., Şubă A. *The solution of the problem of center for cubic differential systems with four invariant straight lines*, Scientific Annals of the "Al. I. Cuza" University (Romania), Mathematics, 1998, **XLIV**, s.I.a, 517–530.

[6] Cozma D. *The problem of the center for cubic systems with two parallel invariant straight lines and one invariant conic*. Nonlinear Differ. Equ. and Appl., 2009, **16**, 213–234.

[7] Cozma D. *The problem of the center for cubic systems with two parallel invariant straight lines and one invariant conic*. Annals of Differential Equations, 2010, **30**, No. 4, 385–399.

[8] Cozma D. *Center problem for cubic systems with a bundle of two invariant straight lines and one invariant conic*. Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2012, No. 1 (68), 32–49.

[9] Cozma D. *Integrability of cubic systems with invariant straight lines and invariant conics*, Chişinău: Ştiinţa, 2013.

[10] Cozma D. *Darboux integrability and rational reversibility in cubic systems with two invariant straight lines*. Electronic Journal of Differential Equations, 2013, **2013**, No. 23, 1–19.

[11] Cozma D. *The problem of the center for cubic systems with two parallel invariant straight lines and one invariant cubic*, ROMAI Journal, 2015, **11**, No. 2, 63–75.

[12] Cozma D. *Center conditions for a cubic differential system with a bundle of two invariant straight lines and one invariant cubic*, ROMAI Journal, 2017, **13**, No. 2 (accepted).

[13] Lloyd N. G., Pearson J. M. *A cubic differential system with nine limit cycles*, Journal of Applied Analysis and Computation, 2012, **2**, No. 3, 293–304.

[14] Lloyd N. G., Pearson J. M. *Centres and limit cycles for an extended Kukles system*, Electronic Journal of Differential Equations, 2007, **2007**, No. 119, 1–23.

[15] Lyapunov A. M. *The general problem of stability of motion*, Gostekhizdat, Moscow, 1950 (in Russian).

[16] Han M., Romanovski V., Zhang X. *Integrability of a family of 2-dim cubic systems with degenerate infinity*, Rom. Journ. Phys., 2016, **61**, Nos. 1–2, 157–166.

[17] Popa M. N., Pricop V. V. *Applications of algebraic methods in solving the center-focus problem*. Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2013, No. 1 (71), 45–71.

[18] Sadovskii A. P., Shcheglova T. V. *Solution of the center-focus problem for a nine-parameter cubic system*. Differential Equations, 2011, **47**, No. 2, 208–223.

[19] Schlomiuk D. *Algebraic and geometric aspects of the theory of polynomial vector fields.* Bifurcations and periodic orbits of vector fields. Kluwer Academic Publishes, 1993, 429–467.

[20] Şubă A. *Partial integrals, integrability and the center problem*, Differential Equations, 1996, **32**, No. 7, 884–892.

[21] Şubă A., Vacaraş O. *Cubic differential systems with an invariant straight line of maximal multiplicity*, Annals of the University of Craiova, Mathematics and Computer Science Series, 2015, **42**, No. 2, 427–449.

[22] Żołądek H. *On certain generalization of the Bautin's theorem*, Nonlinearity, 1994, **7**, 273–279.

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