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Pretorsions in modules and associated closure operators

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Abstract. This article contains the results on the pretorsions of the module category R-Mod and on the closure operators defined by them. The pretorsions of R-Mod can be described in diverse forms: by classes of modules, filters of left ideals of R, closure operators, dense submodules, etc. In the set $\mathbb{PT}$ of pretorsions of R-Mod the main operations are studied, as well as their expressions in terms of classes of modules, filters, operators, etc. The approximations of pretorsions by jansian pretorsions and by torsions are mentioned.

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1 Introduction. Preliminary notions and facts

In this work the pretorsions of a module category R-Mod and the associated closure operators are studied. The main operations in the set $\mathbb{PT}$ of pretorsions of R-Mod are investigated. The multilateral descriptions of pretorsions of R-Mod are accentuated. Pretorsions of R-Mod can be considered as subfunctors of the identity functor of R-Mod (r); as pretorsion classes of R-Mod $(\mathcal{T}_r)$; as filters of left ideals of R $(\mathcal{E}_r)$; as closure operators of the lattice $\mathbb{L}({}_RR)$ of left ideals of R (t_r); as closure operators of the category R-Mod (C^r); as functions defined by dense submodules $(\boldsymbol{\mathcal{F}}_s^r)$.

The main operations in $\mathbb{PR}$ are investigated and the representations of them by corresponding constructions $(\mathcal{T}_r, \mathcal{E}_r, C^r$, etc.) are indicated. For the given pretorsion $r \in \mathbb{PT}$ the least jansian pretorsion or torsion containing r is shown.

Let R be a ring with unit $1 \neq 0$ and R-Mod be the category of unitary left R-modules. A *preradical* r of R-Mod is a subfunctor of identity functor of R-Mod, i.e. $r(M) \subseteq M$ for every $M \in R$-Mod and $f\big(r(M)\big) \subseteq r(M')$ for every R-morphism $f\colon M \to M'$ of R-Mod. A preradical r is *hereditary* (or *pretorsion*) if $r(N) = r(M) \cap N$ for every $N \in \mathbb{L}(M)$, where $\mathbb{L}(M)$ is the lattice of submodules of M [1–4].

We denote by $\mathbb{PR}$ the class of all preradicals of R-Mod, and by $\mathbb{PT}$ the class (set) of all pretorsions of R-Mod. Every preradical $r \in \mathbb{PR}$ defines two classes of modules:

$\mathcal{T}_r = \{M \in R\text{-Mod} \mid r(M) = M\}$ – the class of r-*torsion* modules;

$\mathcal{F}_r = \{M \in R\text{-Mod} \mid r(M) = 0\}$ – the class of r-*torsionfree* modules.

The class $\mathcal{K} \subseteq R$-Mod is called *pretorsion class* if it is closed under homomorphic images and direct sums. If $\mathcal{K} \subseteq R$-Mod is closed under submodules, it is called *hereditary* class. It is well known the following description of pretorsions by classes of modules [1–4].

Proposition 1.1. *There exists a monotone bijection between the pretorsions of R-Mod and* ***hereditary pretorsion classes*** *of R-Mod. It is defined by the rules: $r \rightsquigarrow \mathcal{T}_r,\ \mathcal{T} \rightsquigarrow r^{\mathcal{T}}$, where $r^{\mathcal{T}}(M) = \sum\limits_{\alpha\in\mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid N_\alpha \in \mathcal{T}\}$.*

An important peculiarity of pretorsions consists in the fact that they can be characterized by the special sets of left ideals of R ([1–4]). A set of left ideals $\mathcal{E} \subseteq \mathbb{L}(_RR)$ is called a *preradical filter* (left linear topology, topologizing filter) if the following conditions are satisfied:

(a_1) If $I \in \mathcal{E}$ and $a \in R$, then $(I : a) = \{x \in R \mid xa \in I\} \in \mathcal{E}$;

(a_2) If $I \in \mathcal{E}$ and $I \subseteq J$, $J \in \mathbb{L}(_RR)$, then $J \in \mathcal{E}$;

(a_3) If $I, J \in \mathcal{E}$, then $I \cap J \in \mathcal{E}$.

Proposition 1.2. *There exists a monotone bijection between the pretorsions of R-Mod and the* ***preradical filters*** *of R. It is defined by the mappings:*

$r \rightsquigarrow \mathcal{E}_r = \{I \in \mathbb{L}(_RR) \mid r(R/I) = R/I\}$;

$\mathcal{E} \rightsquigarrow r_{\mathcal{E}},\quad r_{\mathcal{E}}(M) = \{m \in M \mid (0 : m) \in \mathcal{E}\}$.

Remark. From the Proposition 1.2 follows that $\mathbb{PT}$ is a set, in contrast to $\mathbb{PR}$ which in general case is a class.

Therefore investigating the pretorsions we can use the diverse form of their expressions: $r, \mathcal{T}_r, \mathcal{E}_r$. The other three forms of presentation of pretorsions will be indicated in the following account.

2 Operations in the set of pretorsions $\mathbb{PT}$

In the set $\mathbb{PT}$ of pretorsions of R-Mod can be defined the following operations:

– *the meet* $\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha$, where $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\big)(M) = \bigcap\limits_{\alpha\in\mathfrak{A}} r_\alpha(M),\ \{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$;

– *the join* $\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha$, where $\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha = \bigwedge \{s \in \mathbb{PT} \mid s \geq r_\alpha\ \ \forall\ \alpha \in \mathfrak{A}\}$;

– *the product* $r \cdot s$, where $(r \cdot s)(M) = r\big(s(M)\big)$;

– *the coproduct* $r \# s$, where $[(r \# s)(M)]/s(M) = r\big(M/s(M)\big)$.

Remarks. 1. The product $r \cdot s$ of two pretorsions coincides with their meet $r \wedge s$, since using the heredity of r we have:

$$(r \cdot s)(M) = r\,(s(M)) = r(M) \cap s(M) = (r \wedge s)(M).$$

So in continuation we consider the set $\mathbb{PT}\,(\wedge, \vee, \#)$ equipped by three operations, where $\mathbb{PT}\,(\wedge,\vee)$ is a complete lattice.

2. In [1] the operation $(r : s)$ is defined in $\mathbb{PR}$ by the rule $[(r : s)(M)]/r(M) = r\big(M/s(M)\big)$, so $(r\ :\ s) = s\,\#\,r$. Our notation is more convenient and more coordinated with the other notations.

A series of properties of the defined operations are indicated in [1, 4], etc.

Now we will show how can be expressed the operations of $\mathbb{PT}\,(\wedge,\vee, \#)$ by the classes of modules $\mathcal{T}_r$, corresponding to the pretorsions $r \in \mathbb{PT}$. For that we remind that P. Gabriel [5] defined the product $\mathbb{C}\cdot\mathbb{D}$ of two closed (fermeé) classes of modules as follows:

$$\mathbb{C} \cdot \mathbb{D} = \{M \in R\text{-Mod} \mid M/\,\mathbb{D}M \in \mathbb{C}\},$$

where $\mathbb{D}M = \sum\limits_{\alpha\in\mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid N_\alpha \in \mathbb{D}\}$. We will preserve this rule, changing only the notation for hereditary pretorsion classes:

$$\mathcal{T}_r \,\#\, \mathcal{T}_s = \{M \in R\text{-Mod} \mid M/\,s(M) \in \mathcal{T}_r\}.$$

In parallels with the operations in $\mathbb{PT}$, we define the following operations on the classes of modules of the form $\mathcal{T}_r$, where $r \in \mathbb{PT}$:

– *the meet*: $\bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha} = \bigcap\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha}$;

– *the join*: $\bigvee\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha} = \bigcap\,\{\mathcal{T}_s \mid \mathcal{T}_s \supseteq \mathcal{T}_{r_\alpha}\ \ \forall\, \alpha \in \mathfrak{A}\}$;

– *the coproduct*: $\mathcal{T}_r \,\#\, \mathcal{T}_s = \{M \in R\text{-Mod} \mid M/\,s(M) \in \mathcal{T}_r\}$.

Now we indicate the concordance between the operations of $\mathbb{PT}$ and the operations with the hereditary pretorsion classes of R-Mod.

Proposition 2.1. $\mathcal{T}_{\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha}$ *for every family* $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$.

Proof. By the definitions we have:

$$M \in \mathcal{T}_{\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha} \Leftrightarrow \big(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\big)(M) = M \Leftrightarrow \bigcap\limits_{\alpha\in\mathfrak{A}} r_\alpha(M) = M \Leftrightarrow r_\alpha(M) = M \ \ \forall\, \alpha \in \mathfrak{A} \Leftrightarrow$$
$$\Leftrightarrow M \in \mathcal{T}_{r_\alpha} \ \ \forall\, \alpha \in \mathfrak{A} \Leftrightarrow M \in \bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha}. \qquad \square$$

Similarly from the definitions follows the

Proposition 2.2. $\mathcal{T}_{\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha} = \bigvee\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha}$. $\square$

Proposition 2.3. $\mathcal{T}_{r\#\,s} = \mathcal{T}_r \,\#\, \mathcal{T}_s$ *for every pretorsions* $r, s \in \mathbb{PT}$.

Proof. By the definition of coproduct we obtain:

$$M \in \mathcal{T}_{r\#s} \Leftrightarrow (r \,\#\, s)(M) = M \Leftrightarrow [(r \,\#\, s)(M)]/s(M) = M/s(M) \Leftrightarrow$$

$$\Leftrightarrow r\big(M/s(M)\big) = M/s(M) \Leftrightarrow M/s(M) \in \mathcal{T}_r \Leftrightarrow M \in \mathcal{T}_r \,\#\, \mathcal{T}_s. \qquad \square$$

In continuation we will consider the expression of operations of $\mathbb{PT}$ by the corresponding *preradical filters* $\mathcal{E}_r$ of pretorsions $r \in \mathbb{PT}$. Denote $\boldsymbol{\mathfrak{PF}}$ the set of all preradical filters of R and define in this set the following operations:

– *the meet*: $\bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha} = \bigcap\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}$;

– *the join*: $\bigvee\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha} = \bigcap \{\mathcal{E} \in \boldsymbol{\mathfrak{PF}} \mid \mathcal{E} \supseteq \mathcal{E}_{r_\alpha} \ \ \forall\, \alpha \in \mathfrak{A}\}$;

– *the coproduct*: $\mathcal{E}_r \,\#\, \mathcal{E}_s = \{I \in \mathbb{L}({}_RR) \mid \exists\ H \in \mathcal{E}_r,\ I \subseteq H$ such that $(I : a) \in \mathcal{E}_s \ \ \forall\, \alpha \in H\}$.

Remark. The latter operation is defined in [4] by changing the order of terms. Our notation is harmonized with the previous ones.

Now we show the relations between these operations and the operations of $\mathbb{PT}$.

Proposition 2.4. $\mathcal{E}_{\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}$ *for every family* $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$.

Proof follows from the Proposition 2.1. $\square$

Proposition 2.5. $\mathcal{E}_{\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha} = \bigvee\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}$.

Proof follows from the Proposition 2.2. $\square$

Proposition 2.6. $\mathcal{E}_{\,r\#\,s} = \mathcal{E}_r \,\#\, \mathcal{E}_s$ *for every* $r, s \in \mathbb{PT}$.

Proof. $(\subseteq)$ Let $I \in \mathcal{E}_{\,r\#\,s}$. Then from the Proposition 2.3 follows:

$$R/I \in \mathcal{T}_{r\#s} = \mathcal{T}_r \,\#\, \mathcal{T}_s = \{M \in R\text{-Mod} \mid M/s(M) \in \mathcal{T}_r\}.$$

Therefore $(R/I)\,/\,s(R/I) \in \mathcal{T}_r$.

Now we consider the left ideal $H \subseteq R$ defined by the rule $(H/I) = s(R/I)$. Then $(R/I)\,/\,(H/I) \in \mathcal{T}_r$, so $R/H \in \mathcal{T}_r$, i.e. $H \in \mathcal{E}_r$. Moreover, from the definition of H we have $H/I \in \mathcal{T}_s$.

So we have a left ideal $H \in \mathcal{E}_r$, $I \subseteq H$ with the condition $H/I \in \mathcal{T}_s$ (i.e. $(I : a) \in \mathcal{E}_s$ for every $a \in H$). By the definition this means that $I \in \mathcal{E}_r \,\#\, \mathcal{E}_s$.

$(\supseteq)$ Let $I \in \mathcal{E}_r \,\#\, \mathcal{E}_s$, i.e. there exists a left ideal $H \subseteq R$ such that $I \subseteq H$ and $H/I \in \mathcal{T}_s$. Consider the left ideal $H' \subseteq R$ defined by the rule $H'/I = s(R/I)$. From the condition $H/I \in \mathcal{T}_s$ follows that $H/I \subseteq s(R/I) = H'/I$, so $H \subseteq H'$. Since $H \in \mathcal{E}$, now we have $H' \in \mathcal{E}_r$, i.e. $R/H' \in \mathcal{T}_r$.

From the other hand, by Proposition 2.3 and definitions we have:

$$\begin{aligned}\mathcal{E}_{r\# s} &= \big\{I \in \mathbb{L}(_R R) \mid R/I \in \mathfrak{T}_{r\# s} = \mathfrak{T}_r \# \mathfrak{T}_s = \\ &= \{M \in R\text{-Mod} \mid M/s(M) \in \mathfrak{T}_r\}\big\} = \{I \in \mathbb{L}(_R R) \mid (R/I)/s(R/I) \in \mathfrak{T}_r\} = \\ &= \{I \in \mathbb{L}(_R R) \mid (R/I)/(H'/I) \in \mathfrak{T}_r\} = \{I \in \mathbb{L}(_R R) \mid R/H' \in \mathfrak{T}_r\}.\end{aligned}$$

Now from the relation $R/H' \in \mathfrak{T}_r$ obtained above follows that $I \in \mathcal{E}_{r\# s}$. □

3 Pretorsions and closure operators in $\mathbb{L}(_R R)$

In this section we will indicate a new form of expression for pretorsions of R-Mod by some closure operators of the lattice $\mathbb{L}(_R R)$ of left ideals of R. With this intention we consider a mapping $t\colon \mathbb{L}(_R R) \to \mathbb{L}(_R R)$ and the following conditions on t:

1°) $t(I) \supseteq I$ (*extension*);

2°) $t\big(t(I)\big) = t(I)$ (*idempotency*);

3°) $I \subseteq J \Rightarrow t(I) \subseteq t(J)$ (*monotony*);

4°) $t(I:a) = \big(t(I):a\big)\ \ \forall\, a \in R$ (*modularity*);

5°) $t(I \cap J) = t(I) \cap t(J)$ (*linearity*).

It is well known that the conditions 1°) – 3°) define the ordinary notion of *closure operator* of the lattice $\mathbb{L}(_R R)$.

Definition 3.1. If the mapping t satisfies the conditions 1°) – 4°), then it is called the *modular closure operator* of $\mathbb{L}(_R R)$ [3, 6]. If t satisfies the conditions 1°), 3°), 4°), 5°), then it will be called the *modular preclosure operator* of $\mathbb{L}(_R R)$.

There exists a monotone bijection between the *torsions* of R-Mod and the modular closure operators of $\mathbb{L}(_R R)$ [3, 6]. This bijection is obtained as follows:

$$r \rightsquigarrow t_r, \quad t_r(I) = \{a \in R \mid (I:a) \in \mathcal{E}_r\};$$
$$t \rightsquigarrow r_t, \quad r_t(M) = \{m \in M \mid t(0:m) = R\}.$$

Now we will show the generalization of this result for the case of pretorsions [7].

Proposition 3.1. *Let $r \in \mathbb{PT}$ and $\mathcal{E}_r$ be the associated preradical filter. Define the operator t_r of $\mathbb{L}(_R R)$ by the rule:*

$$t_r(I) = \{a \in R \mid (I:a) \in \mathcal{E}_r\}.$$

Then t_r is a modular preclosure operator of $\mathbb{L}(_R R)$.

Proof. Verify the conditions 1°), 3°), 4°), 5°) for t_r.

1°) If $a \in I$, then $(I:a) = R$, $R \in \mathcal{E}_r$, so $a \in t_r(I)$.

3°) If $I \subseteq J$ and $a \in t_r(I)$, then $(I:a) \in \mathcal{E}_r$. From the relation $(I:a) \subseteq (J:a)$ by (a_2) it follows that $(J:a) \in \mathcal{E}_r$, so $a \in t_r(J)$.

4°) By the definitions we have:

$$t_r(I:a) = \{x \in R \mid \big((I:a):x\big) = (I:xa) \in \mathcal{E}_r\};$$
$$\big(t_r(I):a\big) = \{x \in R \mid xa \in t_r(I)\} = \{x \in R \mid (I:xa) \in \mathcal{E}_r\},$$

so 4°) is true.

5°) The expressions of 5°) have the form:

$$t_r(I \cap J) = \{a \in R \mid \big((I \cap J):a\big) \in \mathcal{E}_r\} = \{a \in R \mid (I:a) \cap (J:a) \in \mathcal{E}_r\};$$
$$t_r(I) \cap t_r(J) = \{a \in R \mid (I:a) \in \mathcal{E}_r\} \cap \{a \in R \mid (J:a) \in \mathcal{E}_r\} =$$
$$= \{a \in R \mid (I:a) \cap (J:a) \in \mathcal{E}_r\},$$

therefore 5°) is true. □

Proposition 3.2. *Let t be a modular preclosure operator of $\mathbb{L}({}_RR)$. Define the function r_t by the rule:*

$$r_t(M) = \{m \in M \mid t(0:m) = R\}$$

for every $M \in R$-Mod. Then r_t is a pretorsion of R-Mod.

Proof. It is obvious that the set $r_t(M)$ forms a submodule of M. Moreover, for every R-morphism $f\colon M \to M'$ we have $f\big(r_t(M)\big) = \{f(m) \mid t(0:m) = R\}$. Since $\big(0:f(m)\big) \supseteq (0:m)$, we obtain $t\big(0:f(m)\big) \supseteq t(0:m) = R$, so $t\big(0:f(m)\big) = R$, i.e. $f(m) \in r_t(M')$. Therefore $f\big(r_t(M)\big) \subseteq r_t(M')$ and r_t is a preradical of R-Mod.

Finally, for every $N \in \mathbb{L}(M)$ we have:

$$r_t(M) \cap N = \{n \in N \mid n \in r_t(M)\} = \{n \in N \mid t(0:n) = R\} = r_t(N),$$

so r_t is hereditary, i.e. $r_t \in \mathbb{PT}$. □

Theorem 3.3. *The mappings $r \rightsquigarrow r_t$ and $t \rightsquigarrow r_t$ define a monotone bijection between the pretorsions of R-Mod and the modular preclosure operators of $\mathbb{L}({}_RR)$.*

Proof. Taking into account the Propositions 3.1 and 3.2, it is sufficient to prove that the indicated mappings define a bijection, i.e. $r = r_{t_r}$ and $t = t_{r_t}$.

Verify the first relation:

$$r_{t_r}(M) = \{m \in M \mid t_r(0:m) = R\} = \{m \in M \mid \{a \in R \mid (0:am) \in \mathcal{E}_r\} = R\} =$$
$$= \{m \in M \mid (0:am) \in \mathcal{E}_r \ \forall\, a \in R\} = \{m \in M \mid \big((0:m):a\big) \in \mathcal{E}_r \ \forall\, a \in R\} =$$
$$= \{m \in M \mid (0:m) \in \mathcal{E}_r\} = r(M),$$

so $r = r_{t_r}$.

On the other hand, for every modular preclosure operator t of $\mathbb{L}({}_RR)$ we have:

$$t_{r_t}(I) = \{a \in R \mid (I:a) \in \mathcal{E}_{r_t}\},$$

where $\mathcal{E}_{r_t} = \{I \in \mathbb{L}({}_RR) \mid t(I) = R\}$. Now using the modularity 4°) we obtain:

$$t_{r_t}(I) = \{a \in R \mid t(I:a) = R\} = \{a \in R \mid \big(t(I):a\big) = R\} =$$
$$= \{a \in R \mid a \in t(I)\} = t(I),$$

therefore $t = t_{r_t}$. □

We remark the fact that the preradical filter of a pretorsion r_t has the form $\mathcal{E}_{r_t} = \{I \in \mathbb{L}(_RR) \mid t(I) = R\}$, i.e. it coincides with the set of t-dense left ideals of R.

In continuation we show haw can be obtained from the Theorem 3.3 the similar result for the *torsions*, which was formulated above. We remind that by definition a torsion is a hereditary radical. As the pretorsions, they can be described by the filters of left ideals of R. Supplementing the conditions $(a_1) - (a_3)$ which define the preradical filters (see Section 1), we now consider the following conditions on the set of left ideals $\mathcal{E} \subseteq \mathbb{L}(_RR)$:

(a_4) If $I_\alpha \in \mathcal{E}$, $\alpha \in \mathfrak{A}$, then $\bigcap\limits_{\alpha \in \mathfrak{A}} I_\alpha \in \mathcal{E}$;

(a_5) If $I \subseteq J$, $J \in \mathcal{E}$ and $(I : j) \in \mathcal{E}$ for every $j \in J$, then $I \in \mathcal{E}$.

If $r \in \mathbb{PT}$ and $\mathcal{E}_r$ satisfies the condition (a_4), then r is called *jansian pretorsion*. Such pretorsions will be considered in Section 7.

The set of left ideals $\mathcal{E} \subseteq \mathbb{L}(_RR)$ is called a *radical filter* (Gabriel filter, left Gabriel topology) if it satisfies the conditions $(a_1), (a_2)$ and (a_5). The description of torsions of R-Mod by the radical filters of $\mathbb{L}(_RR)$ consists in the following [1–5].

Proposition 3.4. *The mappings*

$$r \rightsquigarrow \mathcal{E}_r, \quad \mathcal{E}_r = \{I \in \mathbb{L}(_RR) \mid r(R/I) = R/I\};$$
$$\mathcal{E} \rightsquigarrow r_{\mathcal{E}}, \quad r_{\mathcal{E}}(M) = \{m \in M \mid (0 : m) \in \mathcal{E}\}$$

define a monotone bijection between the torsions of R-Mod and radical filters of $\mathbb{L}(_RR)$. $\square$

Now we will indicate the transition from the pretorsions to the torsions of R-Mod in terms of the modular preclosure operators of $\mathbb{L}(_RR)$.

Proposition 3.5. *Let $r \in \mathbb{PT}$ and t_r be the associated modular preclosure operator of $\mathbb{L}(_RR)$. Then the following conditions are equivalent:*

1) *r is a torsion;*

2) *t_r satisfies the condition 2°), i.e. it is idempotent.*

Proof. 1) $\Rightarrow$ 2) If r is a torsion with radical filter $\mathcal{E}_r$, then by the definitions we have:

$$t_r(I) = \{a \in R \mid (I : a) \in \mathcal{E}_r\};$$

$$t_r\big(t_r(I)\big) = \{b \in R \mid \big(t_r(I) : b\big) \in \mathcal{E}_r\}.$$

Let $b \in t_r\big(t_r(I)\big)$. From $I \subseteq t_r(I)$ follows $(I : b) \subseteq \big(t_r(I) : b\big) \in \mathcal{E}_r$. Moreover, for every $d \in \big(t_r(I) : b\big)$ we have $\big((I : b) : d\big) \in \mathcal{E}_r$. Indeed, from $d \in \big(t_r(I) : b\big)$ follows $db \in t_r(I)$, i.e. $(0 : db) \in \mathcal{E}_r$. Therefore $\big((I : b) : d\big) = (I : db) \in \mathcal{E}_r$, so $\big((I : b) : d\big) \in \mathcal{E}_r$.

Now we can use the condition (a_5) in the situation $(I : b) \subseteq \big(t_r(I) : b\big) \in \mathcal{E}_r$, from which follows that $(I : b) \in \mathcal{E}_r$, which means that $b \in t_r(I)$. So we have $t_r\big(t_r(I)\big) \subseteq t_r(I)$, which implies the condition 2°).

2) $\Rightarrow$ 1) Suppose that the operator t_r is idempotent. By the definitions we have:

$$t_r(I) = \{a \in R \mid (I : a) \in \mathcal{E}_r\}, \quad t_r\big(t_r(I)\big) = \{b \in R \mid \big(t_r(I) : b\big) \in \mathcal{E}_r\}.$$

Therefore the idempotence of t_r means that from the $\big(t_r(I) : b\big) \in \mathcal{E}_r$ follows $(I : b) \in \mathcal{E}_r$.

It is sufficient to prove that the filter $\mathcal{E}_r$ satisfies the condition (a_5). Suppose that $I \subseteq J$, $J \in \mathcal{E}_r$ and $(I : j) \in \mathcal{E}_r$ for every $j \in J$. From the last condition we have $J \subseteq t_r(I)$ and from the $J \in \mathcal{E}_r$ we obtain $t_r(I) \in \mathcal{E}_r$, therefore $\big(t_r(I) : b\big) \in \mathcal{E}_r$ for every $b \in R$. By the idempotence of t_r now follows $(I : b) \in \mathcal{E}_r$ for every $b \in R$, therefore $I \in \mathcal{E}_r$. So the condition (a_5) is satisfied for $\mathcal{E}_r$, i.e. r is a torsion. $\square$

Applying Theorem 3.3 and Proposition 3.5, we obtain the mentioned above result on torsions ([3, 6]).

Corollary 3.6. *The mappings $r \rightsquigarrow t_r$ and $t \rightsquigarrow r_t$ define a monotone bijection between the torsions of R-Mod and modular closure operators of $\mathbb{L}({}_R R)$.* $\square$

4 Pretorsions and closure operators of *R*-Mod

An important aspect of pretorsions of R-Mod, closely related by the previous, consists in the description of pretorsions with the help of some *closure operators of the category R-Mod*. We remind firstly the necessary definitions and facts ([8–10]).

A *closure operator* of R-Mod is defined as a function C, which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$ and $M \in R$-Mod, a submodule of M denoted by $C_M(N)$, such that the following conditions are satisfied:

(c_1) $N \subseteq C_M(N)$ (*extension*);

(c_2) $N_1 \subseteq N_2 \Rightarrow C_M(N_1) \subseteq C_M(N_2)$ (*monotony*);

(c_3) $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$ for every R-morphism $f \colon M \to M'$ and $N \subseteq M$ (*continuity*).

We denote by $\mathbb{CO}$ the class of all closure operators of R-Mod. Define in this class the following operations:

– *the meet* $\bigwedge\limits_{\alpha \in \mathfrak{A}} C_\alpha$, where $\big(\bigwedge\limits_{\alpha \in \mathfrak{A}} C_\alpha\big)_M(N) = \bigcap\limits_{\alpha \in \mathfrak{A}} \big[(C_\alpha)_M(N)\big]$;

– *the join* $\bigvee\limits_{\alpha \in \mathfrak{A}} C_\alpha$, where $\big(\bigvee\limits_{\alpha \in \mathfrak{A}} C_\alpha\big)_M(N) = \sum\limits_{\alpha \in \mathfrak{A}} \big[(C_\alpha)_M(N)\big]$;

– *the product* $C \cdot D$, where $\big(C \cdot D\big)_M(N) = C_M\big(D_M(N)\big)$;

– *the coproduct* $C \# D$, where $\big(C \# D\big)_M(N) = C_{D_M(N)}(N)$.

We remind also the main types of closure operators of R-Mod. An operator $C \in \mathbb{CO}$ is called:

– *weakly hereditary*, if $C_M(N) = C_{C_M(N)}(N)$;

– *idempotent*, if $C_M(N) = C_M\big(C_M(N)\big)$;

– *hereditary*, if $C_N(L) = C_M(L) \cap N$, where $L \subseteq N \subseteq M$;

– *cohereditary*, if $\big(C_M(N) + K\big)/K = C_{M/K}\big((N+K)/K\big)$, where $K, N \in \mathbb{L}(M)$;

– *maximal*, if $C_M(N)/N = C_{M/N}(\bar{0})$ $\big($or: $C_M(N)/K = C_{M/K}(N/K)$, where $K \subseteq N \subseteq M\big)$;

– *minimal*, if $C_M(N) = C_M(0) + N$ $\big($or: $C_M(N) = C_M(L) + N$, where $L \subseteq N \subseteq M\big)$.

There exists a close relation between the class of preradicals $\mathbb{PR}$ and the class of closure operators $\mathbb{CO}$ of R-Mod, which is expressed by the following mappings:

1) $\Phi\colon\ \mathbb{CO} \to \mathbb{PR}$, where $\Phi(C) = r_{_C}$, $r_{_C}(M) = C_M(0)$;

2) $\Psi_1\colon\ \mathbb{PR} \to \mathbb{CO}$, where $\Psi_1(r) = C^r$, $[(C^r)_M(N)]/N = r(M/N)$;

3) $\Psi_2\colon\ \mathbb{PR} \to \mathbb{CO}$, where $\Psi_2(r) = C_r$, $(C_r)_M(N) = N + r(M)$.

The class of maximal closure operators $\mathbb{M}ax\,(\mathbb{CO})$ coincides with the operators of the form C^r, $r \in \mathbb{PR}$, and the pair (Φ, Ψ_1) establishes the bijection $\mathbb{M}ax\,(\mathbb{CO}) \cong \mathbb{PR}$. Dually, the class of minimal closure operators $\mathbb{M}in\,(\mathbb{CO})$ coincides with the class of closure operators of the form C_r, $r \in \mathbb{PR}$, and the pair (Φ, Ψ_2) defines a bijection $\mathbb{M}in\,(\mathbb{CO}) \cong \mathbb{PR}$.

In continuation we remind the effect of the defined above mappings to the class $\mathbb{PT}$ of *pretorsions* of R-Mod. The following statements are proved in [9] (Part IV, Propositions 2.7, 3.5).

Proposition 4.1. 1) *The pair of mappings* (Φ, Ψ_1) *defines a monotone bijection between the pretorsions of R-Mod and the maximal and hereditary closure operators of R-Mod.*

2) *The pair* (Φ, Ψ_2) *determines a monotone bijection between the pretorsions of R-Mod and the minimal and hereditary closure operators of R-Mod.* □

Denoting by $\mathbb{M}ax\,(\mathbb{HCO})$ the class of maximal and hereditary closure operators of $\mathbb{CO}$, we have the bijection $\mathbb{PT} \cong \mathbb{M}ax\,(\mathbb{HCO})$.

Let $r \in \mathbb{PT}$ and $\mathcal{E}_r$ be the associated preradical filter. Then the maximal and hereditary closure operator C^r of R-Mod is defined by the rule $[C^r_M(N)]/N = r(M/N)$ and can be expressed by the filter $\mathcal{E}_r$ as follows.

Lemma 4.2. $C^r_M(N) = \{m \in M \mid (N:m) \in \mathcal{E}_r\}$, *where* $(N:m) = \{a \in R \mid am \in N\}$.

Proof. It is obvious that the set $\{m \in M \mid (N:m) \in \mathcal{E}_r\}$ is a submodule of M, containing N. Since

$$r(M/N) = \{m + N \in M/N \mid \big(0 : (m+N)\big) = (N:m) \in \mathcal{E}_r\},$$

by the definition of $C^r_M(N)$ follows the statement. □

For the subsequent investigations we need the following conditions on the closure operator $C \in \mathbb{CO}$:

(c_4) $\big(C_M(N) : m\big) = C_R(N:m)$ for every $N \in \mathbb{L}(M)$ and $m \in M$ (*modularity*);

(c_5) $C_M(N \cap L) = C_M(N) \cap C_M(L)$ for every $N, L \in \mathbb{L}(M)$ (*linearity*).

Proposition 4.3. *Let* $r \in \mathbb{PT}$ *and* C^r *be the respective maximal and hereditary closure operator of R-Mod. Then* C^r *satisfies the conditions* (c_4) *and* (c_5), *i.e. it is modular and linear.*

Proof. (c_4) From the definitions and Lemma 4.2 we have:

$$\big(C^r_M(N) : m\big) = \{a \in R \mid am \in C^r_M(N)\} = \{a \in R \mid (N : am) =$$
$$= \big((N:m):a\big) \in \mathcal{E}_r\},$$
$$C^r_R(N:m) = \{a \in R \mid \big((N:m):a\big) = (N:am) \in \mathcal{E}_r\},$$

so (c_4) is true.

(c_5) The expressions of (c_5) have the form:

$$C^r_M(N \cap L) = \{m \in M \mid \big((N \cap L) : m\big) = (N:m) \cap (L:m) \in \mathcal{E}_r\},$$
$$C^r_M(N) \cap C^r_M(L) = \{m \in M \mid (N:m) \in \mathcal{E}_r\} \cap \{m \in M \mid (L:m) \in \mathcal{E}_r\} =$$
$$= \{m \in M \mid (N:m) \cap (L:m) \in \mathcal{E}_r\}.$$ □

Now we mention the relation of these results with the facts of Section 3. Let $r \in \mathbb{PT}$ with the corresponding closure operator C^r. If we consider the action of C^r on the lattice $\mathbb{L}({}_RR)$ (i.e. we fix $M = {}_RR$), then we obtain a closure operator C^r_R of $\mathbb{L}({}_RR)$.

Corollary 4.4. *If* $r \in \mathbb{PT}$, *then the operator* t_r *of* $\mathbb{L}({}_RR)$ *defined by the rule* $t_r(I) = \{a \in R \mid (I:a) \in \mathcal{E}_r\}$ *coincides with the operator* C^r_R, *therefore* C^r_R *is a modular preclosure operator of* $\mathbb{L}({}_RR)$.

Proof. From the Lemma 4.2 we have $C^r_R(I) = \{a \in R \mid (I:a) \in \mathcal{E}_r\}$, therefore $C^r_R = t_r$. From Proposition 3.1 it now follows that C^r_R is a modular preclosure operator of $\mathbb{L}({}_RR)$. □

Now we show the similar results on the *torsions* of R-Mod. For that we use the following

Lemma 4.5. *Let $r \in \mathbb{PT}$ and C^r be the associated maximal closure operator. Then the following conditions are equivalent:*

1) *r is a torsion;*
2) *C^r is an idempotent closure operator.*

Proof. 1) $\Rightarrow$ 2) If r is a torsion, then $\mathcal{E}_r$ is a radical filter, so it satisfies the condition (a_5). Let $m \in C^r_M\big(C^r_M(N)\big)$. Then $\big(C^r_M(N) : m\big) \in \mathcal{E}_r$ and it is obvious that $(N : m) \subseteq \big(C^r_M(N) : m\big)$. Moreover, for every $a \in \big(C^r_M(N) : m\big)$ we have $am \in C^r_M(N)$, so $(N : am) = \big((N : m) : a\big) \in \mathcal{E}_r$. Now we can apply the condition (a_5) in the situation $(N : m) \subseteq \big(C^r_M(N) : m\big) \in \mathcal{E}_r$, concluding that $(N : m) \in \mathcal{E}_r$, i.e. $m \in C^r_M(N)$. This proves the relation $C^r_M\big(C^r_M(N)\big) \subseteq \big(C^r_M(N)$, which is sufficient for the idempotence of C^r.

2) $\Rightarrow$ 1) If C^r is idempotent, then the operator $C^r_R = t_r$ of $\mathbb{L}({}_RR)$ satisfies the condition 2°), i.e. it is idempotent. From the Proposition 3.5 this is equivalent to the fact that r is a torsion. $\square$

From the Proposition 4.1 and Lemma 4.5 follows the

Corollary 4.6. *The pair of mappings (Φ, Ψ_1) define a monotone bijection between the torsions of R-Mod and maximal, hereditary and idempotent closure operators of R-Mod.* $\square$

It is interesting that the closure operators of the form C^r, where $r \in \mathbb{PT}$ (i.e. maximal and hereditary) can be characterized by the conditions (c_4) and (c_5) indicated above. By Proposition 4.3 every closure operator of such type satisfies the conditions (c_4) and (c_5). Now we show that the inverse statement is also true.

Proposition 4.7. *Let $C \in \mathbb{CO}$ and C satisfies the conditions (c_4) and (c_5), i.e. it is modular and linear. Then the set of C-dense left ideals $\mathcal{E}_C = \{I \in \mathbb{L}({}_RR) \mid C_R(I) = R\}$ is a preradical filter, the pretorsion defined by $\mathcal{E}_C$ coincides with $r_C = \Phi(C)$ and $C = C^{r_C}$.*

Proof. Verify the conditions $(a_1) - (a_3)$ for $\mathcal{E}_C$.

(a_1) If $I \in \mathcal{E}_C$ and $a \in R$, then $C_R(I) = R$ and from (c_4) we have
$$C_R(I : a) = \big(C_R(I) : a\big) = (R : a) = R,$$
therefore $(I : a) \in \mathcal{E}_C$.

(a_2) If $I \in \mathcal{E}_C$ and $I \subseteq J$, then $C_R(I) = R$ and from (c_2) we have $C_R(I) \subseteq C_R(J)$, so $C_R(J) = R$, i.e. $J \in \mathcal{E}_C$.

(a_3) If $I, J \in \mathcal{E}_C$, then $C_R(I) = C_R(J) = R$, so from (c_5) we obtain $C_R(I \cap J) = C_R(I) \cap C_R(J) = R$, i.e. $I \cap J \in \mathcal{E}_C$.

This proves that $\mathcal{E}_C$ is a preradical filter, therefore it defines a pretorsion $r_{\mathcal{E}_C}$. It coincides with $r_C = \Phi(C)$, since from the definitions and (c_4) we have:

$r_{\mathcal{E}_C}(M) = \{m \in M \mid (0:m) \in \mathcal{E}_C\} = \{m \in M \mid C_R(0:m) = R\} =$
$= \{m \in M \mid \big(C_M(0):m\big) = R\} = \{m \in M \mid m \in C_M(0)\} = C_M(0) = r_C(M)$.

The similar arguments show that $C^{r_C} = C$. Indeed, for every $N \subseteq M$ using (c_4) we obtain:

$(C^{r_C})_M(N) = \{m \in M \mid (N:m) \in \mathcal{E}_C\} = \{m \in M \mid C_R(N:m) = R\} =$
$= \{m \in M \mid \big(C_M(N):m\big) = R\} = \{m \in M \mid m \in C_M(N)\} = C_M(N)$. □

From Propositions 4.3 and 4.7 follows the

Corollary 4.8. *The pair of mappings (Φ, Ψ_1) defines a monotone bijection between the pretorsions of R-Mod and the modular and linear closure operators of $\mathbb{CO}$.* □

5 Relations between the operations of $\mathbb{PT}$ and $\mathbb{CO}$

By Proposition 4.1 the pair of mappings (Φ, Ψ_1) defines a monotone bijection $\mathbb{PT} \cong \mathbb{M}ax\,(\mathbb{HCO})$. Now we specify the form of operations in $\mathbb{M}ax\,(\mathbb{HCO})$:

- *the meet*: $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N) = \bigcap\limits_{\alpha\in\mathfrak{A}} \big[\big(C_\alpha\big)_M(N)\big]$;
- *the join*: $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha = \bigwedge \{D \in \mathbb{M}ax\,(\mathbb{HCO}) \mid D \supseteq C_\alpha \ \ \forall\, \alpha \in \mathfrak{A}\}$;
- *the product*: $(C \cdot D)_M(N) = C_M\big(D_M(N)\big)$.

In the case of pretorsions the relation $r \cdot s = r \wedge s$ was mentioned (Section 2). Similarly, in the case of *hereditary* closure operators the coproduct coincides with the meet.

Lemma 5.1. *If $C, D \in \mathbb{CO}$ and C is hereditary, then $C \# D = C \wedge D$.*

Proof. For every $N \subseteq M$ from the heredity of C used in the situation $N \subseteq D_M(N) \subseteq M$ we obtain:

$$(C \# D)_M(N) = C_{D_M(N)}(N) = C_M(N) \cap D_M(N) = (C \wedge D)_M(N). \qquad \square$$

For this reason in the case of hereditary closure operators we consider only three operations: meet, join and product, so we have the bijection: $\mathbb{PT}(\wedge, \vee, \#) \cong \mathbb{M}ax\,(\mathbb{HCO})(\wedge, \vee, \cdot)$. The following statements show the concordance of operations in this bijection.

Proposition 5.2. $C^{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha\in\mathfrak{A}} C^{r_\alpha}$ *for every family* $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$.

Proof. Since $\mathcal{E}_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}$ (Proposition 2.4) we have:

$\Big(C^{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha}\Big)_M(N) = \{m \in M \mid (N:m) \in \mathcal{E}_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha}\}$;

$\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} C^{r_\alpha}\big)_M(N) = \bigcap\limits_{\alpha\in\mathfrak{A}} \big[C_M^{r_\alpha}(N)\big] = \bigcap\limits_{\alpha\in\mathfrak{A}} \big[\{m \in M \mid (N:m) \in \mathcal{E}_{r_\alpha}\}\big] =$
$= \{m \in M \mid (N:m) \in \bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha} = \mathcal{E}_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha}\}$. □

Proposition 5.3. $C^{\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha} = \bigvee\limits_{\alpha\in\mathfrak{A}} C^{r_\alpha}$ *for every family* $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$.

Proof follows from the Proposition 2.5. □

Proposition 5.4. $C^{r\,\#\,s} = C^r \cdot C^s$ *for any pretorsions* $r, s \in \mathbb{PT}$.

Proof. We verify the relation $C_M^{r\,\#\,s}(N) = C_M^r\big(C_M^s(N)\big)$, where $N \subseteq M$.

($\subseteq$) Let $m \in C_M^{r\,\#\,s}(N)$. Then from the Proposition 2.6 and from the definitions we have:

$(N:m) \in \mathcal{E}_{r\,\#\,s} = \mathcal{E}_r \,\#\, \mathcal{E}_s =$

$= \{I \in \mathbb{L}({}_RR) \mid \exists\, H \in \mathcal{E}_r,\ I \in H$ such that $(I:a) \in \mathcal{E}_s\ \ \forall\, a \in H\}$.

So there exists $H \in \mathcal{E}_r$ such that $(N:m) \subseteq H$ and $\big((N:m):a\big) = (N:am) \in \mathcal{E}_s$ for every $a \in H$. Therefore for every element $am + N \in (Hm+N)/N$ we have $\big(0:(am+N)\big) = (N:am) \in \mathcal{E}_s$, which means that $(Hm+N)/N \in \mathcal{T}_s$. But then $(Hm+N)/N \subseteq s(M/N) = C_M^s(N)/N$, so $Hm \subseteq C_M^s(N)$ and $H \subseteq \big(C_M^s(N):m\big)$. Since $H \in \mathcal{E}_r$, now we have $\big(C_M^s(N):m\big) \in \mathcal{E}_r$, which means that $m \in C_M^r\big(C_M^s(N)\big)$.

($\supseteq$) Let $m \in C_M^r\big(C_M^s(N)\big)$. Then $\big(C_M^s(N):m\big) \in \mathcal{E}_r$ and denoting $H = \big(C_M^s(N):m\big)$ we have $H \in \mathcal{E}_r$ and $Hm \subseteq C_M^s(N)$. From the relation $N \subseteq C_M^s(N)$ follows $(N:m) \subseteq \big(C_M^s(N):m\big) = H$. Moreover, for every $a \in H$ we have $am \in C_M^s(N)$, i.e. $(N:am) = \big((N:m):a\big) \in \mathcal{E}_s$. By the definition this means that $(N:m) \in \mathcal{E}_r \,\#\, \mathcal{E}_s = \mathcal{E}_{r\,\#\,s}$, therefore $m \in C_M^{r\,\#\,s}(N)$. □

From the previous statements we conclude that the mapping Ψ_1 preserves the meets and joins, but it converts the coproduct into the product.

6 Characterization of pretorsions by dense submodules

Let $C \in \mathbb{CO}$. For every $M \in R$-Mod we denote:

$\boldsymbol{\mathfrak{F}}_1^C(M) = \{N \in \mathbb{L}(M) \mid C_M(N) = M\}$ – the set of C-*dense* submodules of M;

$\boldsymbol{\mathfrak{F}}_2^C(M) = \{N \in \mathbb{L}(M) \mid C_M(N) = N\}$ – the set of C-*closed* submodules of M.

Thus the operator $C \in \mathbb{CO}$ defines two functions $\boldsymbol{\mathfrak{F}}_1^C$ and $\boldsymbol{\mathfrak{F}}_2^C$, which distinguish in every module M the set of C-dense submodules $\boldsymbol{\mathfrak{F}}_1^C(M)$ and the set of C-closed submodules $\boldsymbol{\mathfrak{F}}_2^C(M)$. In some cases by the help of these functions the operator C can be reestablished. More exactly, C can be restored by $\boldsymbol{\mathfrak{F}}_1^C$ if and only if it is weakly hereditary. Dually, C can be reestablished by $\boldsymbol{\mathfrak{F}}_2^C$ if and only if it is idempotent ([9], Part I).

Now we remind some results on the function $\mathfrak{F}_1^C$ defined by C-dense submodules. For every $C \in \mathbb{CO}$ the function $\mathfrak{F}_1^C$ satisfies the following conditions:

1) If $N \in \mathfrak{F}_1^C(M_\alpha),\ M_\alpha \subseteq M,\ \alpha \in \mathfrak{A},$ then $N \in \mathfrak{F}_1^C\big(\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha\big)$;

2) If $N \subseteq P \subseteq M$ and $N \in \mathfrak{F}_1^C(P)$, then $N + K \in \mathfrak{F}_1^C(P + K)$ for every $K \subseteq M$;

3) If $f : M \to M'$ is an R-morphism and $N \in \mathfrak{F}_1^C(M)$, then $f(N) \in \mathfrak{F}_1^C\big(f(M)\big)$.

An abstract function $\mathfrak{F}$ which separates in every module M a set of submodules $\mathfrak{F}(M)$ is called a *function of type* $\mathfrak{F}_1$, if it satisfies the conditions 1) – 3). Then $\mathfrak{F}$ defines a closure operator $C^{\mathfrak{F}}$ by the rule:

$$(C^{\mathfrak{F}})_M(N) = \sum_{\alpha \in \mathfrak{A}} \{M_\alpha \subseteq M \mid N \in \mathfrak{F}(M_\alpha)\}.$$

The description of the weakly hereditary closure operators by the functions of type $\mathfrak{F}_1$ consists in the following ([9], Part I, Theorem 2.6).

Proposition 6.1. *The mappings $C \rightsquigarrow \mathfrak{F}_1^C$ and $\mathfrak{F} \rightsquigarrow C^{\mathfrak{F}}$ define a monotone bijection between the* ***weakly hereditary*** *closure operators of $\mathbb{CO}$ and the functions of type $\mathfrak{F}_1$ of R-Mod.*

By the restriction of this bijection we obtain the similar result for the *hereditary* closure operators of $\mathbb{CO}$. For that the following condition on the abstract function $\mathfrak{F}$ is considered:

(Her) If $N \subseteq P \subseteq M$ and $N \in \mathfrak{F}(M)$, then $N \in \mathfrak{F}(P)$.

Proposition 6.2. *The mappings $C \rightsquigarrow \mathfrak{F}_1^C$ and $\mathfrak{F} \rightsquigarrow C^{\mathfrak{F}}$ define a monotone bijection between the* ***hereditary*** *closure operators of $\mathbb{CO}$ and the abstract functions of type $\mathfrak{F}_1$ of R-Mod, which satisfy the condition* (Her) ([9], *Part* II, *Corollary* 2.3).

In a similar way from the Proposition 6.1 the description of *weakly hereditary and maximal* closure operators can be obtained. With this aim the following condition on a function $\mathfrak{F}$ is considered:

(Max) If $K \subseteq N \subseteq M$ and $N/K \in \mathfrak{F}(M/K)$, then $N \in \mathfrak{F}(M)$.

Proposition 6.3. *The mappings $C \rightsquigarrow \mathfrak{F}_1^C$ and $\mathfrak{F} \rightsquigarrow C^{\mathfrak{F}}$ define a monotone bijection between the* ***weakly hereditary and maximal*** *closure operators of $\mathbb{CO}$ and the abstract functions of type $\mathfrak{F}_1$, which satisfy the condition* (Max) ([9], *Part* II, *Corollary* 3.3).

From Propositions 6.2 and 6.3 we have

Corollary 6.4. *The mappings $C \rightsquigarrow \mathfrak{F}_1^C$ and $\mathfrak{F} \rightsquigarrow C^{\mathfrak{F}}$ establish a monotone bijection between the* ***hereditary and maximal*** *closure operators of $\mathbb{CO}$ and the abstract functions of type $\mathfrak{F}_1$, which satisfy the conditions* (Her) *and* (Max).

Now we can use the fact that the pretorsions of R-Mod are described by the maximal and hereditary closure operators of R-Mod, since by Proposition 4.1 we have the bijection: $\mathbb{PT} \cong \mathbb{M}ax(\mathbb{HCO})$. In one's turn the operators of $\mathbb{M}ax(\mathbb{HCO})$ by Corollary 6.4 can be characterized by the abstract functions of type $\mathfrak{F}_1$ with the conditions (Max) and (Her). Therefore the followihg is true.

Proposition 6.5. *There exists a monotone bijection between the pretorsions of R-Mod and the abstract functions of type $\mathfrak{F}_1$, which satisfy the conditions* (Max) *and* (Her)*.*

This bijection has the form:

$r \rightsquigarrow \mathfrak{F}_1^r$, where $\mathfrak{F}_1^r(M) = \{N \in \mathbb{L}(M) \mid (N : m) \in \mathcal{E}_r \ \ \forall\, m \in M\}$;

$\mathfrak{F} \rightsquigarrow r_{\mathfrak{F}}$, where $r_{\mathfrak{F}}(M) = \sum\{M_\alpha \in \mathbb{L}(M) \mid 0 \in \mathfrak{F}(M_\alpha)\}$.

We mention also the fact that for every pretorsion $r \in \mathbb{PT}$ we have $\mathfrak{F}_1^r({}_RR) = \mathcal{E}_r$.

From the exposed above results follows that every pretorsion $r \in \mathbb{PT}$ can be described not only by the class $\mathcal{T}_r$ and the filter $\mathcal{E}_r$, but also by the operator t_r of $\mathbb{L}({}_RR)$, by the operator C^r of R-Mod and by the function $\mathfrak{F}_1^r$, which selects the dense submodules.

7 On some approximations of pretorsions

Concluding this work, we mention some simple methods of approximations of pretorsions by *jansian pretorsions* and by *torsions* of R-Mod. By approximations we means the constructions of the least jansian pretorsion or of the least torsion, which contains the given pretorsion.

Let $r \in \mathbb{PT}$. We denote $L_r = \cap\ \{I_\alpha \in \mathbb{L}({}_RR) \mid I_\alpha \in \mathcal{E}_r\}$. Then L_r is an ideal of R and it is called the *kernel* of r. The following conditions for $r \in \mathbb{PT}$ are equivalent ([1, 3, 4]):

1) r is jansian $\big($see condition (a_4), Section 3$\big)$;

2) $L \in \mathcal{E}_r$;

3) the class $\mathcal{T}_r$ is closed under products: if $M_\alpha \in \mathcal{T}_r$ $(\alpha \in \mathfrak{A})$, then $\prod\limits_{\alpha\in\mathfrak{A}} M_\alpha \in \mathcal{T}_r$.

If r is a jansian pretorsion, then $\mathcal{E}_r = \{I \in \mathbb{L}({}_RR) \mid I \supseteq L_r\}$.

There exists an *antimonotone bijection* between the jansian pretorsions of R-Mod and two sided ideals of R. It is defined by the rules:

$$r \rightsquigarrow L_r, \quad I \rightsquigarrow \mathcal{E}_I = \{I_\alpha \in \mathbb{L}({}_RR) \mid I_\alpha \supseteq I\}.$$

It is obvious that if the pretorsion $r \in \mathbb{PT}$ is jansian, then the associated maximal and hereditary closure operator C^r acts as follows: $C^r_M(N) = \{m \in M \mid (N : m) \supseteq L_r\}$.

It is easy to show how can be expressed by C^r the condition that the pretorsion $r \in \mathbb{PT}$ is jansian. For that we consider the following condition to an arbitrary $C \in \mathbb{CO}$:

(c_6) $C_M\big(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha\big) = \bigcap\limits_{\alpha\in\mathfrak{A}} C_M(N_\alpha)$ for every family $\{N_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{L}(M)$ (*complete linearity*).

Proposition 7.1. *For every $r \in \mathbb{PT}$ the following conditions are equivalent:*

1) *r is a jansian pretorsion;*

2) *the closure operator C^r satisfies the condition (c_6).*

Proof. 1) $\Rightarrow$ 2) If r is jansian, then:

$$m \in C^r_M\big(\bigcap_{\alpha\in\mathfrak{A}} N_\alpha\big) \Leftrightarrow \big((\bigcap_{\alpha\in\mathfrak{A}} N_\alpha) : m\big) \supseteq L_r \Leftrightarrow \bigcap_{\alpha\in\mathfrak{A}} (N_\alpha : m) \supseteq L_r \Leftrightarrow$$
$$\Leftrightarrow m \in \bigcap_{\alpha\in\mathfrak{A}} C^r_M(N_\alpha), \text{ so is true } (c_6).$$

2) $\Rightarrow$ 1) If C^r is complete linear, then $C^r_R\big(\bigcap\limits_{I_\alpha\in\mathcal{E}_r} I_\alpha\big) = \bigcap\limits_{I_\alpha\in\mathcal{E}_r} [C^r_R(I_\alpha)] = R$, so $\bigcap\limits_{I_\alpha\in\mathcal{E}_r} I_\alpha = L_r \in \mathcal{E}_r$, i.e. r is jansian. □

Let $r \in \mathbb{PT}$ and L_r be the kernel of the pretorsion r. Then the ideal L_r defines a jansian pretorsion $\hat{r}$, determined by the preradical filter $\mathcal{E}_{\hat{r}} = \{I \in \mathbb{L}(_RR) \mid I \supseteq L_r\}$, i.e. $\hat{r}(M) = \{m \in M \mid (0 : m) \supseteq L_r\}$ for every $M \in R$-Mod.

Proposition 7.2. *$\hat{r}$ is the least jansian pretorsion containing the pretorsion $r \in \mathbb{PT}$.*

Proof. Since $\mathcal{E}_r \subseteq \mathcal{E}_{\hat{r}}$, we have $r \leq \hat{r}$ and $\hat{r}$ is a jansian pretorsion with the kernel L_r. If $s \in \mathbb{PT}$ is jansian and $r \leq s$, than $\mathcal{E}_r \leq \mathcal{E}_s$, so $L_r \supseteq L_s$, therefore $\hat{r} \leq s$. This means that $\hat{r}$ is the least jansian pretorsion containing r. □

Taking into account this property, $\hat{r}$ is called the *jansian hull* of the pretorsion $r \in \mathbb{PT}$ [4]. For an ideal I of R we denote by r_I the jansian pretorsion defined by I, so that $r_I(M) = \{m \in M \mid (0 : m) \supseteq I\}$.

Proposition 7.3. $\bigwedge\limits_{\alpha\in\mathfrak{A}} \hat{r}_\alpha = r_{\sum\limits_{\alpha\in\mathfrak{A}} L_{r_\alpha}}$ *for every family $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$.*

Proof. We compare the respective preradical filters:

$$\mathcal{E}_{\bigwedge\limits_{\alpha\in\mathfrak{A}} \hat{r}_\alpha} = \bigcap_{\alpha\in\mathfrak{A}} \mathcal{E}_{\hat{r}_\alpha} = \{I \in \mathbb{L}(_RR) \mid I \in \mathcal{E}_{\hat{r}_\alpha} \;\; \forall\, \alpha \in \mathfrak{A}\} =$$
$$= \{I \in \mathbb{L}(_RR) \mid I \supseteq L_{r_\alpha} \;\; \forall\, \alpha \in \mathfrak{A}\} = \{I \in \mathbb{L}(_RR) \mid I \supseteq \sum_{\alpha\in\mathfrak{A}} L_{r_\alpha}\} = \mathcal{E}_{r_{\sum\limits_{\alpha\in\mathfrak{A}} L_{r_\alpha}}}.$$ □

In continuation we show the other type of approximation of a pretorsion $r \in \mathbb{PT}$, namely by the help of *torsions*. Every pretorsion $r \in \mathbb{PT}$ is accompanied by two classes of modules:

$$\mathcal{T}_r = \{M \in R\text{-Mod} \mid r(M) = M\}, \quad \mathcal{F}_r = \{M \in R\text{-Mod} \mid r(M) = 0\}.$$

It is well known that the class $\mathcal{T}_r$ uniquely reestablishes the pretorsion r, while the class $\mathcal{F}_r$ not always determines r.

To clarify the situation it is convenient to use the following operators of "orthogonality", which act to the abstract classes of modules $\mathcal{K} \subseteq R$-Mod ([1–3]):

$$\mathcal{K}^{\uparrow} = \{X \in R\text{-Mod} \mid Hom_R(X, Y) = 0 \;\; \forall\, Y \in \mathcal{K}\},$$

$$\mathcal{K}^{\downarrow} = \{Y \in R\text{-Mod} \mid Hom_R(X, Y) = 0 \;\; \forall\, X \in \mathcal{K}\}.$$

For every $\mathcal{K} \subseteq R$-Mod the class $\mathcal{K}^{\uparrow}$ is a *torsion class* (i.e. it is closed under homomorphic image, direct sums and extensions), and $\mathcal{K}^{\downarrow}$ is a *torsionfree class* (i.e. it is closed under submodules, direct products and extensions). Moreover, $\mathcal{K}^{\downarrow\uparrow}$ is the least torsion class containing $\mathcal{K}$, and $\mathcal{K}^{\uparrow\downarrow}$ is the least torsionfree class containing $\mathcal{K}$. If r is an idempotent radical, then $\mathcal{T}_r = \mathcal{F}_r^{\uparrow}$ and $\mathcal{F}_r = \mathcal{T}_r^{\downarrow}$. In this case $\mathcal{T}_r$ is hereditary if and only if $\mathcal{F}_r$ is stable and this means that r is a torsion.

Lemma 7.4. *If r is a pretorsion, then the class $\mathcal{F}_r = \mathcal{T}_r^{\downarrow}$ is closed under submodules, direct products, extensions and injective envelopes, i.e. $\mathcal{F}_r$ is a torsionfree stable class.*

Proof. The first three properties of the class $\mathcal{F}_r = \mathcal{T}_r^{\downarrow}$ are obvious, since every class of the form $\mathcal{K}^{\downarrow}$ is torsionfree. We verify the stability of $\mathcal{F}_r$: $M \in \mathcal{F}_r$ implies $E(M) \in \mathcal{F}_r$, where $E(M)$ is the injective envelope of M.

Let $M \in \mathcal{F}_r$, i.e. $r(M) = \{m \in M \mid (0 : m) \in \mathcal{E}_r\} = 0$. Suppose that $r\big(E(M)\big) \neq 0$. Then there exists an element $0 \neq x \in E(M)$ such that $(0 : x) \in \mathcal{E}_r$. Since $Rx \neq 0$, we have $Rx \cap M \neq 0$, so there exists an element $0 \neq m = ax \in M$, where $a \in R$, for which $(0 : m) = (0 : ax) = \big((0 : x) : a\big) \in \mathcal{E}_r$, therefore $0 \neq m \in r(M)$, contradiction. This shows that $r\big(E(M)\big) = 0$, i.e. $E(M) \in \mathcal{F}_r$ and the class $\mathcal{F}_r$ is stable. $\square$

Now we remind the relation between the torsions r of R-Mod and the associated classes $\mathcal{T}_r$ and $\mathcal{F}_r$ ([1–3, 6]).

Lemma 7.5. 1) *The mappings $r \rightsquigarrow \mathcal{T}_r$, and $\mathcal{T} \rightsquigarrow r^{\mathcal{T}}$, where $r^{\mathcal{T}}(M) = \sum\limits_{\alpha \in \mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid N_\alpha \in \mathcal{T}\}$, define a monotone bijection between the torsions of R-Mod and the hereditary torsion classes of R-Mod.*

2) *The mappings $r \rightsquigarrow \mathcal{F}_r$, and $\mathcal{F} \rightsquigarrow r_{\mathcal{F}}$, where $r_{\mathcal{F}}(M) = \bigcap\limits_{\alpha \in \mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid M/N_\alpha \in \mathcal{F}\}$, establish an antimonotone bijection between the torsions of R-Mod and the stable torsionfree classes of R-Mod.*

Let $r \in \mathbb{PT}$. By the Lemma 7.4 the class $\mathcal{F}_r$ is a stable torsionfree class, so by the Lemma 7.5 $\mathcal{F}_r$ defines a *torsion* $\widetilde{r}$ such that $\mathcal{T}_{\widetilde{r}} = \mathcal{F}_r^{\uparrow} = \mathcal{T}_r^{\downarrow\uparrow}$ and $\mathcal{F}_{\widetilde{r}} = \mathcal{F}_r$, i.e.

$$\widetilde{r}(M) = \bigcap_{\alpha \in \mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid M/N_\alpha \in \mathcal{F}_r\}.$$

Proposition 7.6. *Let $r \in \mathbb{PT}$. Then the torsion $\widetilde{r}$, defined by the class $\mathcal{F}_r$, is the least torsion containing r.*

Proof. By the definitions the class of modules $\mathcal{T}_{\widetilde{r}} = \mathcal{F}_r^{\uparrow} = \mathcal{T}_r^{\downarrow\uparrow}$ is the least hereditary torsion class, which contains $\mathcal{T}_r$. Therefore $\widetilde{r}$ is the least torsion containing r. □

The torsion $\widetilde{r}$ constructed above is called the *torsion hull* of the pretorsion $r \in \mathbb{PT}$. Then $\mathcal{E}_{\widetilde{r}}$ is the least *radical filter* of R, containing the preradical filter $\mathcal{E}_r$. It is obvious that class of modules $\mathcal{T}_{\widetilde{r}}$ can be directly described by the class $\mathcal{T}_r$, as well as the radical filter $\mathcal{E}_{\widetilde{r}}$ can be expressed by $\mathcal{E}_r$. For example: $\mathcal{E}_{\widetilde{r}} = \{I \in \mathbb{L}({}_RR) \mid \forall\ J \supset I,\quad J \neq R,\quad \exists\, a \notin J \text{ such that } (J : a) \in \mathcal{E}_r\}$ ([2], Chapter VI, Proposition 5.4).

In particular, for the pretorsion $\mathbb{Z}$ defined by the preradical filter of *essential* left ideals $\mathcal{E}_{\mathbb{Z}} = \{I \in \mathbb{L}({}_RR) \mid I \subseteq' {}_RR\}$, the corresponding torsion hull is $\mathbb{Z}_2$ with the radical filter (*Goldie topology*):

$\mathcal{E}_{\mathbb{Z}_2} = \{I \in \mathbb{L}({}_RR) \mid \exists\ J \in \mathcal{E}_{\mathbb{Z}} \text{ such that } I \subset J \text{ and } (I : b) \in \mathcal{E}_{\mathbb{Z}}\ \ \forall\ b \neq J\}$ ([2], Chapter VI, Proposition 6.3).

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