BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(78), 2015, Pages 42–60
ISSN 1024–7696

Relation between Levinson center, chain recurrent set and center of Birkhoff for compact dissipative dynamical systems

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Abstract. In this paper we prove the analogues of Birkhoff's theorem for one-sided dynamical systems (both with continuous and discrete times) with noncompact space having a compact global attractor. The relation between Levinson center, chain recurrent set and center of Birkhoff is established for compact dissipative dynamical systems.

Mathematics subject classification: 37B25, 37B35 37B55, 37L15, 37L30, 37L45.
Keywords and phrases: Global attractors; Birkhoff's center; chain recurrent set.

1 Introduction

Let X be a compact metric space, $(X, \mathbb{R}, \pi)$ be a flow on X, $M \subseteq X$ be a nonempty compact and invariant subset of X. Denote $\Omega(M) := \{x \in M :$ there exist $\{x_n\} \subset M$ and $\{t_n\} \subset \mathbb{R}$ such that $x_n \to x$, $t_n \to +\infty$ as $n \to \infty$ and $\pi(t_n, x_n) \to x\}$. Recall that the point $x \in X$ is called Poisson stable if $x \in \omega_x \bigcap \alpha_x$, where by ω_x (respectively, α_x) the ω (respectively, α)-limits set of x is denoted. The following result is well known (see, for example, [1, 14]).

Theorem 1 (Birkhoff's theorem). *The following statements hold:*

1. *there exists a nonempty, compact and invariant subset* $\mathfrak{B}(\pi) \subseteq X$ *with the properties:*
   (i) $\Omega(\mathfrak{B}(\pi)) = \mathfrak{B}(\pi)$;
   (ii) $\mathfrak{B}(\pi)$ *is the maximal compact invariant subset of* J *with the property* (i).
2. $\mathfrak{B}(\pi) = \overline{\mathcal{P}(\pi)}$, *i. e., the set of all Poisson stable points* $\mathcal{P}(\pi)$ *of the dynamical system* $(X, \mathbb{R}, \pi)$ *is dense in* $\mathfrak{B}(\pi)$.

Remark 1. 1. The set $\mathfrak{B}(\pi)$ is called the Bikkhoff center of dynamical system $(X, \mathbb{R}, \pi)$.

2. Note that Birkhoff theorem remains true also for the discrete dynamical systems $(X, \mathbb{Z}, \pi)$. This fact was established in the work of V. S. Bondarchuk and V. A. Dobrynsky [1].

3. The second statement of Theorem 1 remains true if we replace the center of Birkhoff $\mathfrak{B}(\pi)$ by arbitrary compact invariant set $M \subseteq J$ with the property $\Omega(M) = M$. Namely the following equality takes place: $M = \overline{\mathcal{P}(\pi) \bigcap M}$.

The main result of this paper is the proof of the analogues of Birkhoff theorem for the one-sided dynamical systems (both with continuous and discrete times) with noncompact phase space having a compact global attractor.

2 Birkhoff center

Definition 1. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be:

1. pointwise dissipative if there exists a nonempty compact subset $K \subseteq X$ such that
$$\lim_{t\to+\infty} \rho(\pi(t,x), K) = 0 \tag{1}$$
for all $x \in X$;

2. compactly dissipative if there exists a nonempty compact subset $K \subseteq X$ such that (1) holds uniformly with respect to x on every compact subset from X.

Remark 2. Every compact dissipative dynamical system is pointwise dissipative. The converse, generally speaking, is not true (see, for example, [4, Ch.I]).

Theorem 2 (see [4, Ch.I])**.** *Suppose that $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system, then there exists a nonempty, compact, invariant subset $J \subseteq X$ possessing the following properties:*

1. J attracts every compact subset A from X, i. e.,
$$\lim_{t\to+\infty} \rho(\pi(t,x), J) = 0$$
uniformly with respect to $x \in A$;

2. J is orbitally stable, i.e., for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\rho(\pi(t,x), J) < \varepsilon$ for all $t \geq 0$;

3. J is the maximal compact invariant subset of X.

Let M be a positively invariant and closed subset of X. Denote by $J^+_x(M) := \{p \in X :$ there exist $\{x_n\} \subseteq M$ and $t_n \to +\infty$ such that $x_n \to x$ and $\pi(t_n, x_n) \to p$ as $n \to +\infty\}$.

Lemma 1. *Let M be a positively invariant and closed subset of X. If $p_n \to p$, $x_n \to x$ as $n \to \infty$ and $p_n \in J^+_{x_n}(M)$, then $p \in J^+_x(M)$.*

Proof. Let ε be an arbitrary positive number, $p_n \to p$ and $x_n \to x$ as $n \to \infty$. Then there exists a number $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\rho(p_n, p) < \varepsilon/3 \text{ and } \rho(x_n, x) < \varepsilon/3$$

for all $n \geq n_0$. Since $p_n \in J^+_{x_n}(M)$ for all $n \in \mathbb{N}$, then there exist $\{x_n^m\} \subseteq M$ and $\{t_n^m\}$ (for all $m \in \mathbb{N}$) such that $x_n^m \to x_n$, $t_n^m \to +\infty$ and $\pi(t_n^m, x_n^m) \to p_n$ as $m \to \infty$. In particular, for given ε there exists $n < m_n = m_n(\varepsilon) \in \mathbb{N}$ such that

$$\rho(x_n^m, x_n) < \varepsilon/3 \text{ and } \rho(\pi(t_n^m, x_n^m), p_n) < \varepsilon/3$$

for all $m \geq m_n$. Denote by $\bar{x}_n := x_n^{m_n}$ and $\bar{t}_n := t_n^{m_n} > n$. Note that $\{\bar{x}_n\} \subseteq M$, $\bar{t}_n \to +\infty$ as $n \to \infty$ and

$$\rho(\bar{x}_n, x) = \rho(x_n^{m_n}, x) \leq \rho(x_n^{m_n}, x_n) + \rho(x_n, x) < \varepsilon/3 + \varepsilon/3 < \varepsilon$$

for all $n \geq n_0(\varepsilon)$, i.e., $\bar{x}_n \to x$ as $n \to \infty$. In addition we have

$$\rho(\pi(\bar{t}_n, \bar{x}_n), p) = \rho(\pi(t_n^{m_n}, x_n^{m_n}), p) \leq \rho(\pi(t_n^{m_n}, x_n^{m_n}), p_n) + \rho(p_n, p) < \varepsilon/3 + \varepsilon/3 < \varepsilon$$

for all $n \geq n_0$. Thus for the point p we find the sequence $\{\bar{x}_n\} \subseteq M$ and $\bar{x}_n \to +\infty$ as $n \to \infty$ such that $\bar{x}_n \to x$ and $\pi(\bar{t}_n, \bar{x}_n) \to p$ as $n \to \infty$, i. e., $p \in J^+_x(M)$. Lemma is proved. $\square$

Lemma 2. *Let M be a positively invariant and closed subset of X and $x \in X$. The following statements hold:*

1. *$J^+_x(M) \subseteq M$ for all $x \in M$;*
2. *the set $J^+_x(M)$ is closed and positively invariant;*
3. *if M is compact, then $J^+_x(M)$ is invariant.*

Proof. Let $p \in J^+_x(M)$ and $t \in \mathbb{T}$, then there are $\{x_n\}$ and $t_n \to +\infty$ such that $x_n \to x$ and $\pi(t_n, x_n) \to p$ as $n \to \infty$. Then we have $\pi(t,p) = \lim_{n\to\infty} \pi(t, \pi(t_n, x_n)) = \lim_{n\to\infty} \pi(t + t_n, x_n)$ and, consequently, $\pi(t,p) \in J^+_x(M)$ because $x_n \in M$ and M is closed and positively invariant. Finally, it is evident that $J^+_x(M) \subseteq M$ for all $x \in M$.

Now we will establish the second statement of Lemma. Let $\{p_n\}$ be a sequence from $J^+_x(M)$ such that $p_n \to p$ as $n \to \infty$, then $p_n \in J^+_{x_n}(M)$ where $x_n := x$ for all $n \in \mathbb{N}$. By Lemma 1 $p \in J^+_x(M)$ because $p_n \to p$ and $x_n \to x$ as $n \to \infty$. Let us show now that the set $J^+_x(M)$ is positively invariant. Indeed, let $t \in \mathbb{T}$ and $p \in J^+_x(M)$, then there are $\{x_n\} \subseteq M$ and $t_n \to +\infty$ as $n \to \infty$ such that $\pi(t_n, x_n) \to p$ as $n \to \infty$. Note that $\pi(t,p) = \lim_{n\to\infty} \pi(t + t_n, x_n)$ and, consequently, $\pi(t,p) \in J^+_x(M)$.

Suppose that the set M is compact and $p \in J^+_x(M)$, then there are $\{x_n\} \subseteq M$ and $t_n \to +\infty$ as $n \to \infty$ such that $\pi(t_n, x_n) \to p$ as $n \to \infty$. Let $t \in \mathbb{T}$ be an arbitrary number, then for sufficiently large $n \in \mathbb{N}$ we have $t_n - t \in \mathbb{T}$ because $t_n \to +\infty$ as $n \to \infty$. Since the set M is positively invariant and compact, then without loss of generality we can suppose that the sequence $\{\pi(t_n - t, x_n)\}$ is convergent. Denote by p_t its limit, then we obtain $p = \lim\limits_{n\to\infty} \pi(t_n - t + t, x_n) = \lim\limits_{n\to\infty} \pi(t, \pi(t_n - t, x_n)) = \pi(t, p_t)$ and, consequently, $p \in \pi(t, J^+_x(M))$, i. e., $J^+_x(M) \subseteq \pi(t, J^+_x(M))$ for all $t \in \mathbb{T}$. Thus $J^+_x(M)$ is positively and negatively invariant, i.e., it is invariant. ☐

Definition 2. Let M be a subset of X. A point $x \in X$ is said to be non-wandering with respect to M if $x \in J^+_x(M)$.

Denote by $\Omega(M) := \{x \in M : \ x \in J^+_x(M)\}$ the set of all non-wandering points of M with respect to M.

Remark 3. Let A and B be two closed and positively invariant subsets of X, then $\Omega(A) \subseteq \Omega(B)$.

Definition 3. A point $p \in X$ is said to be:

– Poisson stable in the positive direction if $x \in \omega_x$;

– Poisson stable in the negative direction if there exists an entire trajectory $\gamma_x \in \Phi_x$ such that $x \in \alpha_{\gamma_x}$, where $\alpha_{\gamma_x} := \{q \in X :$ there exists $t_n \to -\infty$ such that $\gamma_x(t_n) \to q$ as $n \to \infty\}$;

– Poisson stable if it is Poisson stable in the both directions.

Lemma 3. *Let M be a nonempty, closed and positively invariant set, then the following statements hold:*

1. *the set $\Omega(M)$ is closed;*
2. *if $p \in M$ is Poisson stable in the positive direction, then $p \in \Omega(M)$;*
3. *if the point $p \in M$ and $\gamma \in \Phi_p$ is an entire trajectory such that $\gamma(\mathbb{S}) \subset M$ and $p \in \alpha_\gamma$, then $p \in \Omega(M)$.*

Proof. The first statement directly follows from Lemma 1 and definition of $\Omega(M)$.

Let $p \in M$ and $p \in \omega_p$, then there exists a sequence $t_n \to +\infty$ such that $\pi(t_n, p) \to p$ as $n \to \infty$. Let $p_n := p$ for all $n \in \mathbb{N}$, then $p_n \to p$ and $\pi(t_n, p_n) \to p$ as $n \to \infty$. This means that $p \in J^+_p(M)$, i.e., $p \in \Omega(M)$.

Let $p \in M$, $\gamma \in \Phi_p$, $\gamma(\mathbb{S}) \subset M$ and $p \in \alpha_\gamma$. Then there exists a sequence $t_n \to +\infty$ such that $\gamma(-t_n) \to p$ as $n \to \infty$. Denote by $p_n := \gamma(-t_n)$, then $p_n \to p$ and $p = \pi(t_n, p_n) \to p$ as $n \to \infty$. Thus $p \in J^+_p(M)$ and, consequently, $p \in \Omega(M)$. ☐

Lemma 4. *Suppose that M is a nonempty, compact positively invariant set and $\mathcal{M}$ is a nonempty, compact minimal subset of M, then $\mathcal{M} \subseteq \Omega(M)$.*

Proof. Let $p \in \mathcal{M}$ and $\gamma \in \Phi_p$ be an entire trajectory of $(X, \mathbb{T}, \pi)$ passing through p at the initial moment such that $\gamma(\mathbb{S}) \subseteq M$. Since $\mathcal{M}$ is minimal, ω_p and α_γ are nonempty, compact and invariant we have $\alpha_\gamma = \omega_p = \mathcal{M}$. In particular there exists a sequence $\tau_n \to +\infty$ such that $p_n := \gamma(-\tau_n) \to p$ as $n \to \infty$. Note that $\pi(\tau_n, p_n) = p$ for all $n \in \mathbb{N}$ and, consequently, $p \in \Omega(\mathcal{M}) \subseteq \Omega(M)$. ☐

Corollary 1. *If M is a nonempty, compact positively invariant set, then $\Omega(M) \neq \emptyset$.*

Proof. Let M be a nonempty, compact and positively invariant set of $(X, \mathbb{T}, \pi)$. By Birkhoff theorem there exists a nonempty minimal subset $\mathcal{M} \subseteq M$ and by Lemma 4 we have $\mathcal{M} \subseteq \Omega(M)$. □

Denote by Φ_x the set of all entire trajectories γ_x of $(X, \mathbb{T}, \pi)$ passing through the point x at the initial moment $t = 0$.

Lemma 5. *Suppose that M is a nonempty, compact and positively invariant set. Then $\Omega(M)$ is a nonempty, compact and positively invariant subset of M.*

Proof. By Corollary 1 the set $\Omega(M)$ is a nonempty subset. By Lemma 1 the set $\Omega(M)$ is closed. Since $\Omega(M) \subseteq M$ and M is compact, then $\Omega(M)$ is so. Let now $p \in \Omega(M)$ and $t \in \mathbb{T}$, then there are $p_n \to p$ ($p_n \in M$) and $t_n \to +\infty$ as $n \to \infty$ such that $p = \lim\limits_{n\to\infty} \pi(t_n, p_n)$. Note that $\pi(t,p) = \lim\limits_{n\to\infty} \pi(t, \pi(t_n, p_n)) = \lim\limits_{n\to\infty} \pi(t_n, \pi(t, p_n))$ and, consequently, $\pi(t,p) \in J^+_{\pi(t,p)}(M)$ because $\lim\limits_{n\to\infty} \pi(t,p_n) = \pi(t,p)$ and $\{\pi(t,p_n)\} \subseteq M$. This means that $\pi(t,p) \in \Omega(M)$, i. e., $\Omega(M)$ is positively invariant. □

Lemma 6. *Let M be a nonempty positively invariant subset of X, then the following statements hold:*

1. *if $(X, \mathbb{T}, \pi)$ is pointwise dissipative, then $\Omega(M)$ is nonempty, closed and positively invariant;*
2. *if the dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative and J is its Levinson center, then the set $\Omega(M)$ is nonempty, compact, positively invariant and $\Omega(M) \subseteq J$;*
3. *if the dynamically system $(X, \mathbb{T}, \pi)$ is point dissipative (but not compactly dissipative), then the set $\Omega(X)$, generally speaking, is not compact.*

Proof. Since $(X, \mathbb{T}, \pi)$ is pointwise dissipative, then $\Omega_M := \overline{\bigcup\{\omega_x : x \in M\}} \subseteq X$ is a nonempty compact invariant subset of $(X, \mathbb{T}, \pi)$ and by Birkhoff's theorem in Ω_M there exists at least one compact minimal subset $\mathcal{M} \subseteq \Omega \subseteq X$. By Corollary 1 $\Omega(M) \neq \emptyset$. Let us show that $\Omega(M)$ is closed. If $p = \lim\limits_{n\to\infty} p_n$ and $p_n \in \Omega(M)$, then $p_n \in J^+_{p_n}(M)$. By Lemma 1 we have $p \in J^+_p(M)$, i.e., $p \in \Omega(M)$. If $p \in \Omega(M)$ and $t \in \mathbb{T}$, then there are $p_n \in M$ and $t_n \to +\infty$ such that $p = \lim\limits_{n\to\infty} \pi(t_n, p_n)$ and, consequently, $\pi(t,p) = \lim\limits_{n\to\infty} \pi(t, \pi(t_n, p_n)) = \lim\limits_{n\to\infty} \pi(t_n, \pi(t, p_n))$, i. e., $\pi(t,p) \in J^+_{\pi(t,p)}(M)$ because $\lim\limits_{n\to\infty} \pi(t,p_n) = \pi(t,p)$. This means that $\pi(t,p) \in \Omega(M)$, i. e., $\Omega(M)$ is positively invariant.

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and $x \in \Omega(M)$, then there exist $\{x_n\} \subseteq M$ and $t_n \to +\infty$ such that $x_n \to x$ and $\pi(t_n, x_n) \to x$ as $n \to \infty$. Denote $K_0 := \overline{\{x_n\}}$, where by bar the closure in X is denoted. Then we have

$$\rho(\pi(t_n, x_n), J) \leq \sup_{p \in K_0} \rho(\pi(t_n, p), J), \tag{2}$$

where J is Levinson center of $(X, \mathbb{T}, \pi)$. Passing to limit in (2) we obtain $x \in J$. By the first item the set $\Omega(X)$ is nonempty, compact and positively invariant.

To prove the third item it is sufficient to construct an example with the corresponding properties. To this end we note that in the works [5] and [8] a dynamical system $(X, \mathbb{T}, \pi)$ with the following properties was constructed:

1. $(X, \mathbb{T}, \pi)$ is point dissipative, but it is not compactly dissipative;
2. $\Omega(X)$ is an unbounded set and, consequently, it is not compact.

Lemma is proved. □

Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system and J be its Levinson center and $M \subseteq X$ be a nonempty, closed and positively invariant subset from X. Denote by $M_1 := \Omega(M)$ the set of all non-wandering (with respect to M) points of $(X, \mathbb{T}, \pi)$. By Lemma 6 the set M_1 is a nonempty, compact and positively invariant subset of J. We denote by $M_2 := \Omega(M_1) \subseteq M_1$ the set of all non-wandering (with respect to M_1) points. By Corollary 1 and Lemma 5 the set M_2 is nonempty, compact and positively invariant. Analogously we define the set $M_3 := \Omega(M_2) \subseteq M_2$ which is also a nonempty, compact and positively invariant set. We can continue this process and we will obtain $M_n := \Omega(M_{n-1})$ for all $n \in \mathbb{N}$. Thus we have a sequence $\{M_n\}_{n \in \mathbb{N}}$ possessing the following properties:

1. for all $n \in \mathbb{N}$ the set M_n is nonempty, compact and positively invariant;
2. $J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots$.

Denote by $M_\lambda := \bigcap\limits_{n=1}^{\infty} M_n$, then M_λ is a nonempty, compact (since the set J is compact) and invariant subset of J. Now we define the set $M_{\lambda+1} := \Omega(M_\lambda)$ and we can continue this process to obtain the following sequence

$$\begin{gathered} J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq \\ M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq M_{\lambda+1} \supseteq \ldots \supseteq M_{\lambda+k} \supseteq \ldots . \end{gathered}$$

Now construct the set $M_\mu := \bigcap\limits_{k=1}^{\infty} M_{\mu+k}$ and we denote by $M_{\mu+1} := \Omega(M_\mu)$ and so on. Thus we will obtain a transfinite sequence of nonempty, compact and positively invariant subsets

$$\begin{gathered} J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq \\ M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\mu \supseteq \ldots . \end{gathered} \tag{3}$$

Since J is a nonempty compact set, then in the sequence (3) there is at most a countable family of different elements, i.e., there exists a γ such that $M_{\nu+1} = M_\nu$.

Definition 4. The set $\mathfrak{B}(M) := M_\nu$ is said to be the center of Birkhoff for the closed and positively invariant set M. If $M = X$, then the set $\mathfrak{B}(\pi) := \mathfrak{B}(X)$ is said to be the Birkhoff center of compact dissipative dynamical system $(X, \mathbb{T}, \pi)$.

Let $(X,\mathbb{T},\pi)$ be a compact dissipative dynamical system and J be its Levinson center. Denote $P(\pi) := \{p \in X : \ p \in \omega_x\}$, then by Lemma 3 we have $P(\pi) \subseteq \mathfrak{B}(\pi) \subseteq J$.

Let K be a nonempty subset of X. Denote by $C(\mathbb{T},K)$ the set of all continuous mappings $f : \mathbb{T} \mapsto K$ equipped with the compact-open topology.

Lemma 7. *Let $(X,\mathbb{T},\pi)$ be a compact dissipative dynamical system and $\mathfrak{B}(\pi)$ be its Birkhoff center. Then the following statements hold:*

1. *$\mathfrak{B}(\pi)$ is a nonempty, compact and invariant set;*
2. *$\mathfrak{B}(\pi)$ is a maximal compact invariant subset M of X such that $\Omega(M) = M$.*

Proof. By Lemma 6 $\mathfrak{B}(\pi)$ is a nonempty, compact and positively invariant set. To finish the proof of the first statement it is sufficient to establish that the set $\mathfrak{B}(\pi)$ is negatively invariant, i.e., $\mathfrak{B}(\pi) \subset \pi(t,\mathfrak{B}(\pi))$ for all $t \in \mathbb{T}$. To this end it is sufficient to show that for all $x \in \mathfrak{B}(\pi)$ the set of all entire trajectories γ_x of $(X,\mathbb{T},\pi)$ passing through the point x at the initial moment with the condition $\gamma_x(\mathbb{S}) \subseteq \mathfrak{B}(\pi)$ is nonempty. Let $x \in \mathfrak{B}(\pi)$. Since $\Omega(\mathfrak{B}(\pi)) = \mathfrak{B}(\pi)$, then there are $\{x_n\} \subseteq \mathfrak{B}(\pi)$ and $\{\tau_n\} \subseteq \mathbb{T}$ such that $x_n \to x$, $\tau_n \to +\infty$ and $\pi(\tau_n,x_n) \to x$. Denote by γ_n the function from $C(\mathbb{S},\mathfrak{B}(\pi))$ defined by the equality $\gamma_n(t) = \pi(t+\tau_n,x_n)$ for all $t \ge -\tau_n$ and $\gamma_n(t) = x_n$ for all $t \le -\tau_n$. We will show that the sequence $\{\gamma_n\}$ is relatively compact in $C(\mathbb{S},\mathfrak{B}(\pi))$. Let $l > 0$. Since the set $\mathfrak{B}(\pi)$ is compact, then it is sufficient to check that the sequence $\{\gamma_n\}$ is equi-continuous on the interval $[-l,l]$. If we suppose that it is not true then there exist $\varepsilon_0 > 0$, $\delta_n \to$ and $t_n^1, t_n^2 \in [-l,l]$ such that

$$|t_n^1 - t_n^2| < \delta_n \ \text{ and } \ \rho(\gamma_n(t_n^1),\gamma_n(t_n^2)) \ge \varepsilon_0 \tag{4}$$

for all $n \in \mathbb{N}$. Without loss of generality we may consider that the sequence $\{\gamma_n(-l)\}$ is convergent and denote its limit by $\bar{x}$. From inequality (4) we have

$$\varepsilon_0 \le \rho(\gamma_n(t_n^1),\gamma_n(t_n^1)) = \rho(\pi(l+t_n^1,\gamma_n(-l)),\pi(l+t_n^2,\gamma_n(-l))). \tag{5}$$

Passing to limit in inequality (5) as $n \to \infty$ and taking into consideration (4), we obtain $\varepsilon_0 \le \rho(\pi(l+\bar{t},\bar{x}),\pi(l+\bar{t},\bar{x})) = 0$, where $\bar{t} := \lim\limits_{n\to\infty} t_n^1 = \lim\limits_{n\to\infty} t_n^2$. The obtained contradiction proves our statement. Thus the sequence $\{\gamma_n\}$ is equi-continuous on $[-l,l]$ and the set $\cup_{n=1}^{\infty}\gamma_n([-l,l]) \subseteq \mathfrak{B}(\pi)$ is relatively compact. Taking into account that l is an arbitrary positive number we conclude that the sequence $\{\gamma_n\}$ is relatively compact in $C(\mathbb{S},\mathfrak{B}(\pi))$. We may suppose that the sequence $\{\gamma_n\}$ is convergent. Denote by $\gamma := \lim\limits_{n\to\infty}\gamma_n$, then $\gamma(0) = x := \lim\limits_{n\to\infty}\pi(\tau_n,x_n)$ and $\gamma \in \Phi_x$ such that $\gamma(\mathbb{S}) \subseteq \mathfrak{B}(\pi) = \Omega(\mathfrak{B}(\pi))$, because by construction $\gamma_n(\mathbb{S}) \subseteq \mathfrak{B}(\pi)$ for all $n \in \mathbb{N}$.

Let now $M \subseteq X$ be an arbitrary nonempty, compact and invariant subset of X with the property $\Omega(M) = M$. Then by construction of $\mathfrak{B}(M)$ we have $\mathfrak{B}(M) = M$. On the other hand $M \subseteq J$, where J is the Levinson center of the compact dissipative dynamical system $(X,\mathbb{T},\pi)$ and, consequently, $\mathfrak{B}(M) \subseteq \mathfrak{B}(X) = \mathfrak{B}(\pi)$. Lemma is completely proved. ◻

Definition 5. Recall that the mapping $f : X \mapsto X$ is said to be open if for all $p \in X$ and $\delta > 0$ the set $f(B(p,\delta))$ is open.

Let $p \in \mathfrak{B}(\pi)$ and $\varepsilon > 0$. Denote by $\tilde{B}(p,\varepsilon) := B(p,\varepsilon) \bigcap \mathfrak{B}(\pi)$.

Lemma 8. *Let $(X,\mathbb{T},\pi)$ be a compact dissipative dynamical system and $\mathfrak{B}(\pi)$ be its Birkhoff center. Then the following statements hold:*

1. *for all $p \in \mathfrak{B}(\pi)$, $\varepsilon > 0$ and $t_0 \in \mathbb{T}$ there exists a number $t = t(p,\varepsilon,t_0) > t_0$ such that $\pi(t,\tilde{B}(p,\varepsilon)) \bigcap \tilde{B}(p,\varepsilon) \neq \emptyset$;*

2. *for all $\varepsilon > 0$, $L > 0$ and $p \in \mathfrak{B}(\pi)$ there are $q \in \tilde{B}(p,\varepsilon)$, $\delta = \delta(L,\varepsilon) > 0$ and $t > L$ such that*
$$\tilde{B}(q,\delta) \bigcup \pi(t,\tilde{B}(q,\delta)) \subset \tilde{B}(p,\varepsilon).$$

Proof. Suppose that under the conditions of Lemma the first statement is not true. Then there exist $p_0 \in \mathfrak{B}(\pi)$, $\varepsilon_0 > 0$ and $t_0 \in \mathbb{T}$ such that

$$\pi(t,\tilde{B}(p_0,\varepsilon_0)) \bigcap \tilde{B}(p_0,\varepsilon_0) = \emptyset \tag{6}$$

for all $t \geq t_0$. On the other hand since $p_0 \in \mathfrak{B}(\pi)$, then there exist $\{p_n\} \subseteq \mathfrak{B}(\pi)$ and $t_n \to +\infty$ such that $\pi(t_n,p_n) \to p$ as $n \to \infty$ and, consequently,

$$\pi(t_n,\tilde{B}(p,\varepsilon_0)) \bigcap \tilde{B}(p,\varepsilon_0) \neq \emptyset \tag{7}$$

for all $n \in \mathbb{N}$. Conditions (6) and (7) are contradictory. The obtained contradiction proves our statement.

Now we will establish the second statement. Let $\varepsilon > 0$, $L > 0$ and $p \in \mathfrak{B}(\pi)$. Since $p \in J^+_p(\mathfrak{B}(\pi))$, then there are $q \in \tilde{B}(p,\varepsilon)$ and $t > L$ such that $\pi(t,q) \in \tilde{B}(p,\varepsilon)$. Let μ be a positive number such that $\tilde{B}(\pi(t,q),\mu) \subset \tilde{B}(p,\varepsilon)$. By continuity of the map $\pi(t,\cdot) : \mathfrak{B}(\pi) \mapsto \mathfrak{B}(\pi)$ there exists a positive number $\delta = \delta(t,q,\varepsilon)$ such that $\tilde{B}(q,\delta) \subset \tilde{B}(p,\varepsilon)$ and $\pi(t,\tilde{B}(q,\delta)) \subset \tilde{B}(\pi(t,q),\mu) \subset \tilde{B}(p,\varepsilon)$. □

Lemma 9. *Suppose that $(X,\mathbb{T},\pi)$ is a dynamical system and the following conditions hold:*

1. *the space X is compact;*

2. *X is an invariant set, i. e., $\pi(t,X) = X$ for all $t \in \mathbb{T}$;*

3. *$\Omega(X) = X$.*

Then for all $x \in X$, $\varepsilon > 0$ and $l > 0$ there exists a number $t > l$ such that

$$\pi^{-t}B(x,\varepsilon) \bigcap B(x,\varepsilon) \neq \emptyset.$$

Proof. Let $x \in X$ and l, ε be two arbitrary positive numbers. Since $x \in J_x^+$, then there are sequences $\{x_n\} \subseteq X$ and $\{t_n\} \subseteq \mathbb{T}$ such that

$$x_n \to x,\ t_n \to +\infty \ \text{ and } \ \pi(t_n, x_n) \to x \tag{8}$$

as $n \to \infty$. For the sufficiently large $n \in \mathbb{N}$ we have

$$t_n > l \ \text{ and } \ x_n, \pi(t_n, x_n) \in B(x, \varepsilon). \tag{9}$$

Let $\gamma_n \in \Phi_{\pi(t_n,x_n)}$ be a full trajectory of $(X, \mathbb{T}, \pi)$ passing through $\pi(t_n, x_n)$ at the initial moment $t = 0$ such that $\gamma_n(s) = \pi(s + t_n, x_n)$ for all $s \geq -t_n$. Then $\gamma_n(-t_n) = x_n \in B(x, \varepsilon)$ and $x_n = \gamma_n(-t_n) \in \pi^{-t_n}(x_n) \subseteq \pi^{-t_n} B(x, \varepsilon)$. Thus we will have

$$x_n \in \pi^{-t_n} B(x, \varepsilon) \bigcap B(x, \varepsilon) \neq \emptyset \tag{10}$$

for all sufficiently large $n \in \mathbb{N}$. □

Corollary 2. *Under the conditions of Lemma 9 for all $x \in X$, $\varepsilon > 0$ and $l > 0$ there exists $t > l$ such that $B(x, \varepsilon) \bigcap \pi^t B(x, \varepsilon) \neq \emptyset$.*

Proof. By Lemma 9 for all $x \in X$, $\varepsilon > 0$ and $l > 0$ there exists $t > l$ such that $\pi^{-t} B(x, \varepsilon) \bigcap B(x, \varepsilon) \neq \emptyset$ and, consequently,

$$\pi^t(\pi^{-t} B(x, \varepsilon) \bigcap B(x, \varepsilon)) \subseteq B(x, \varepsilon) \bigcap \pi^t B(x, \varepsilon) \neq \emptyset.$$

□

Corollary 3. *Suppose that the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and $\mathfrak{B}(\pi)$ is its Birkhoff's center, then for all $x \in \mathfrak{B}(\pi)$, $\varepsilon > 0$ and $l > 0$ there exists a number $t > l$ such that $\pi^{-t} \tilde{B}(x, \varepsilon) \bigcap \tilde{B}(x, \varepsilon) \neq \emptyset$.*

Proof. This statement directly follows from Lemmas 7 and 9. □

Theorem 3. *Suppose that $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system, for all $t > 0$ the mapping $\tilde{\pi}(t, \cdot) := \pi(t, \cdot)\big|_{\mathfrak{B}(\pi)}$ is open, then the set of all Poisson stable in the positive direction points of $(X, \mathbb{T}, \pi)$ is dense in $\mathfrak{B}(\pi)$, i. e., $\mathfrak{B}(\pi) = \overline{P(\pi)}$.*

Proof. By Lemma 3 we have $P(\pi) \subseteq \mathfrak{B}(\pi)$ and, consequently, $\overline{P(\pi)} \subseteq \mathfrak{B}(\pi)$. To finish the proof of Theorem it is sufficiently to show that $\overline{P(\pi)} \supseteq \mathfrak{B}(\pi)$.

Let $p \in \mathfrak{B}(\pi)$ and ε be an arbitrary (sufficient small) positive number. Let $\{t_n\}$ be an increasing sequence such that $\tau_n \to +\infty$. By Lemma 8 (item 2) there exists $t_1 > \tau_1$ such that

$$\tilde{B}[x_1, \varepsilon_1] \subseteq \tilde{B}[p, \varepsilon] \ \text{ and } \ \pi(t_1, \tilde{B}[x_1, \varepsilon_1]) \subseteq \tilde{B}[p, \varepsilon].$$

Since the mapping $\pi(t_1, \cdot)$ is open, then we can choose $x_1 \in \mathfrak{B}(\pi)$ and $\varepsilon_1 > 0$ such that

$$\tilde{B}[x_1, \varepsilon_1] \subset \pi(t_1, \tilde{B}[p, \varepsilon]) \subseteq \tilde{B}[p, \varepsilon].$$

By Lemma 8 there is $t_2 > \tau_2$ such that we will have

$$\tilde{B}[x_2, \varepsilon_2] \subseteq \tilde{B}[x_1, \varepsilon_1] \text{ and } \pi(t_2, \tilde{B}[x_2, \varepsilon_2]) \subseteq \tilde{B}[x_1, \varepsilon_1].$$

Since the mapping $\pi(t_2, \cdot)$ is open we can again choose $x_2 \in \mathfrak{B}(\pi)$ and $0 < \varepsilon_2 < \varepsilon_1/2$ such that

$$\tilde{B}[x_3, \varepsilon_3] \subseteq \tilde{B}[x_2, \varepsilon_2] \text{ and } \pi(t_3, \tilde{B}[x_3, \varepsilon_3]) \subseteq \tilde{B}[x_2, \varepsilon_2].$$

Reasoning analogously we can construct sequences $\{x_n\} \subseteq \mathfrak{B}(\pi)$ and $\{\varepsilon_n\}$ such that $\varepsilon_n < \varepsilon_{n-1}/2$, $\tilde{B}[x_n, \varepsilon_n] \subset \tilde{B}[x_{n-1}, \varepsilon_{n-1}]$ and $\pi(t_n, \tilde{B}[x_n, \varepsilon_n]) \subseteq \tilde{B}[x_{n-1}, \varepsilon_{n-1}]$ for all $n \in \mathbb{N}$, where $\varepsilon_0 := \varepsilon$ and $x_0 := p$. Since $\mathfrak{B}(\pi)$ is a nonempty compact set, then $\Lambda := \bigcap\limits_{n=0}^{\infty} \tilde{B}(x_n, \varepsilon_n) \neq \emptyset$ and it consists of a unique point. Let $\{x\} = \Lambda$. We will show that the point x is Poisson stable in the positive direction. In fact, if $L > 0$ is a sufficiently large number and $\delta > 0$, respectively, sufficiently small number, then we choose a natural number $m \in \mathbb{N}$ with the condition that $t_m > L$ and $\varepsilon_m < \delta$, then $\pi(t_n, \tilde{B}[x_n, \varepsilon_n]) \subseteq \tilde{B}[x_m, \varepsilon_m] \subseteq \tilde{B}[x, \delta]$ for all $n > m$. In particular $\pi(t_n, x) \in \tilde{B}[x, \delta]$ for all $n > m$, i.e., $x \in \omega_x$. Thus $x \in \tilde{B}(p, \varepsilon)$ and, consequently, $\mathfrak{B}(\pi) \subseteq \overline{P(\pi)}$. Theorem is proved. □

Remark 4. 1. Note that the mappings $\tilde{\pi}(t, \cdot)$ $(t \in \mathbb{T})$ are open, if on $\mathfrak{B}(\pi)$ the dynamical system $(X, \mathbb{T}, \pi)$ is invertible, i.e., for all $t \in \mathbb{T}$ the mapping $\tilde{\pi}(t, \cdot) : \mathfrak{B}(\pi) \mapsto \mathfrak{B}(\pi)$ is a homeomorphism.

2. If the dynamical system $(X, \mathbb{T}, \pi)$ is invertible on $\mathfrak{B}(\pi)$, then by Theorem 1.14 [14, Ch.III] (see also Proposal 1.1 from [1], where the analogue of Theorem 1.4 for the discrete dynamical systems was proved) in the set $\mathfrak{B}(\pi)$ the set of all Poisson stable (both in the positive and negative directions) points from X is dense.

Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system. Recall that a compact set $M \subseteq X$ is called a weak attractor of the dynamical system $(X, \mathbb{T}, \pi)$ if $\omega_x \cap M \neq \emptyset$ for all $x \in X$. In this section we establish the relationship between weak attractors of the dynamical system $(X, \mathbb{T}, \pi)$ and its Levinson center.

Theorem 4 (see [4, Ch.I]). *Let $(X, \mathbb{T}, \pi)$ be compactly dissipative, J be its Levinson center and M be a compact weak attractor of the dynamical system $(X, \mathbb{T}, \pi)$. Then $J = J^+(M)$.*

Denote by $J^+_x := \{p \in X :$ there exist the sequences $x_n \to x$ and $t_n \to +\infty$ such that $\pi(t_n, x_n) \to p$ as $n \to \infty\}$ and $J^+(M) := \bigcup\{J^+_x : x \in M\}$.

Lemma 10. *Let $M \subseteq X$ be a nonempty, compact, positively invariant and minimal subset of X. Then the following statements hold:*

1. *the set M is invariant, i.e., $\pi(t, M) = M$ for all $t \in \mathbb{T}$;*

2. *for every $x \in M$ each full trajectory $\gamma \in \Phi_x$ is Poisson stable, i.e., $x \in \omega_x = \alpha_\gamma$.*

Proof. Let $t_0 \in \mathbb{T}$ and $M' := \pi(t_0, M)$, then $M' \subseteq M$ and $\pi(t, M') = \pi(t + t_0, M) \subseteq M$. Since M is a nonempty, compact and positively invariant set, then the set M' is so. Taking into consideration that M is a minimal set we conclude that $M = \pi(t_0, M)$ for all $t_0 \in \mathbb{T}$ and, consequently, it is invariant.

Let now $x \in M$ be an arbitrary point from M, then ω_x is a nonempty, compact and positively invariant subset of M. Since the set M is minimal, then we have $\omega_x = M$. Let now $\gamma \in \Phi_x$ be an arbitrary full trajectory of $(X, \mathbb{T}, \pi)$ with the properties: $\gamma(0) = x$ and $\gamma(\mathbb{S}) \subseteq M$, then its α-limit set $\alpha_\gamma \subseteq M$ is a nonempty and compact subset of $\omega_x = M$. If $p \in \alpha_\gamma$, then there exists a sequence $s_n \to -\infty$ such that $p = \lim_{n\to\infty} \gamma(s_n)$. For all $t \in \mathbb{T}$ the sequence $\{\gamma(t+s_n)\} \subseteq M$ is relatively compact and, consequently, without loss of generality, we may suppose that $\{\gamma(t + s_n)\}$ converges. Denote by p_t its limit, i.e., $p_t := \lim_{n\to\infty} \gamma(t + s_n)$. Note that

$$\pi(t, p) = \lim_{n\to\infty} \pi(t, \gamma(s_n)) = \lim_{n\to\infty} \gamma(t + s_n) \in \alpha_\gamma \subseteq M$$

for all $t \in \mathbb{T}$ and, consequently, ω_p is a nonempty, compact, positively invariant subset of M. On the other hand we have $\omega_p \subseteq \alpha_\gamma \subseteq M$. Since the set M is minimal, then we obtain $M = \omega_p \subseteq \alpha_\gamma \subseteq M$ and, consequently, $\alpha_\gamma = M$. Thus we have $x \in \omega_x = \alpha_\gamma = M$. Lemma is completely proved. □

Theorem 5. *Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system, J be its Levinson center and $\mathfrak{B}(\pi)$ be the Birkhoff center of $(X, \mathbb{T}, \pi)$. Then the following equality takes place: $J = J^+(\mathfrak{B}(\pi))$.*

Proof. By Lemmas 3 and 6 we have $\overline{\mathcal{P}(\pi)} \subseteq \mathfrak{B}(\pi) \subseteq J$ and $\overline{\mathcal{P}(\pi)}$ is a nonempty and compact subset of J. It is not difficult to show that the set $\mathcal{P}(\pi)$ is a weak attractor for $(X, \mathbb{T}, \pi)$. In fact, let $x \in X$ be an arbitrary point of X. Since the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative, then the ω-limit set ω_x of the point x is a nonempty, compact and positively invariant subset of X. By theorem of Birkhoff in ω_x there exists a nonempty, compact, positively invariant and minimal subset $M \subseteq \omega_x$. By Lemma 10 every point p from M is Poisson stable and, consequently, $M \subseteq \mathcal{P}(\pi) \subseteq \overline{\mathcal{P}(\pi)} \subseteq \mathfrak{B}(\pi)$. Thus we have $M \subseteq \omega_x \bigcap \mathfrak{B}(\pi)$ for each $x \in X$, i.e., $\mathfrak{B}(\pi)$ is a weak attractor of $(X, \mathbb{T}, \pi)$. Now to finish the proof of Theorem it is sufficient to apply Theorem 4. □

3 Chain recurrent motions

Let $\Sigma \subseteq X$ be a compact positively invariant set, $\varepsilon > 0$ and $t > 0$.

Definition 6. The collection $\{x = x_0, x_1, x_2, \dots, x_k = y; t_0, t_1, \dots, t_k\}$ of the points $x_i \in \Sigma$ and the numbers $t_i \in \mathbb{T}$ such that $t_i \geq t$ and $\rho(x_i t_i, x_{i+1}) < \varepsilon$ ($i = 0, 1, \dots, k-1$) is called (see, for example, [2,3,6,7,12] and the bibliography therein) a (ε, t, π)-chain joining the points x and y.

Remark 5. Without loss of generality we can suppose always that $t_i \leq 2t$, where t_i and t the numbers figuring in Definition 6 (see, for example, [2, Ch.I]).

We denote by $P(\Sigma)$ the set $\{(x,y) : x,y \in \Sigma, \forall\ \varepsilon > 0\ \forall\ t > 0\ \exists\ (\varepsilon,t,\pi)$-chain joining x and $y\}$. The relation $P(\Sigma)$ is closed, invariant and transitive [2, 6, 10–12].

Definition 7. The point $x \in \Sigma$ is called chain recurrent (in Σ) if $(x,x) \in P(\Sigma)$.

We denote by $\mathfrak{R}(\Sigma)$ the set of all chain recurrent (in Σ) points from Σ.

Remark 6. Note that if Σ_1 and Σ_2 are two positively invariant subsets of $(X,\mathbb{T},\pi)$ with condition $\Sigma_1 \subseteq \Sigma_2$, then $\mathfrak{R}(\Sigma_1) \subseteq \mathfrak{R}(\Sigma_2)$.

Definition 8. Let $A \subseteq X$ be a nonempty positively invariant set. The set A is called (see, for example, [9]) internally chain recurrent if $\mathfrak{R}(A) = A$, and internally chain transitive if the following stronger condition holds: for any $a, b \in A$ and any $\varepsilon > 0$ and $t > 0$, there is an (ε,t,π)-chain in A connecting a and b.

The set of all chain recurrent points for $(X,\mathbb{T},\pi)$ is denoted by $\mathfrak{R}(\Sigma)$, i. e., $\mathfrak{R}(\Sigma) := \{x \in \Sigma : (x,x) \in P(\Sigma)\}$. On $\mathfrak{R}(\Sigma)$ we will introduce a relation $\sim$ as follows: $x \sim y$ if and only if $(x,y) \in P(\Sigma)$ and $(y,x) \in P(\Sigma)$. It is easy to check that the introduced relation $\sim$ on $\mathfrak{R}(\Sigma)$ is a relation of equivalence and, consequently, it is easy to decompose it into the classes of equivalence $\{\mathfrak{R}_\lambda : \lambda \in \mathcal{L}\}$ (internally chain transitive subsets), i. e., $\mathfrak{R}(\Sigma) = \sqcup\{\mathfrak{R}_\lambda :\ \lambda \in \mathcal{L}\}$. By Proposal 2.6 from [2] (see also [6] and [10–12] for the semi-group dynamical systems) the defined above components of the decomposition of the set $\mathfrak{R}(\Sigma)$ are closed and positively invariant.

Lemma 11 (see [9])**.** *Let $x \in X$ and $\gamma \in \Phi_x$. The ω-limit (respectively, α-limit) set of positive (respectively, negative) pre-compact orbit of the point x is internally chain transitive, i. e., $\mathfrak{R}(\omega_x) = \omega_x$ (respectively, $\mathfrak{R}(\alpha_\gamma) = \alpha_\gamma$).*

Let $(X,\mathbb{T},\pi)$ be a compact dissipative dynamical system and J be its Levinson center. Denote by $\mathfrak{R}(\pi) := \mathfrak{R}(J)$.

Problem. Suppose that $(X,\mathbb{T},\pi)$ is a compact dissipative dynamical system and J is its Levinson center. To prove that $\mathfrak{R}(\pi) = \mathfrak{R}(X)$ or to construct a corresponding counterexample.

Remark 7. In the connection with the Problem formulated above it is interesting to note that in the works [5, 8] an example of dynamical system $(X,\mathbb{T},\pi)$ is constructed which posses the following properties:

1. $(X,\mathbb{T},\pi)$ is point dissipative;
2. $(X,\mathbb{T},\pi)$ is asymptotically compact;
3. $(X,\mathbb{T},\pi)$ is not compact dissipative;
4. $\mathfrak{R}(X)$ is an unbounded subset of X.

Denote by $C(\mathbb{T}\times X, X)$ the set of all continuous functions $\pi : \mathbb{T}\times X \mapsto X$ equipped with the compact-open topology. If $K \subset X$ is a compact subset from X, then we denote by

$$d_K(f,g) := \sup_{L>0} \min\{ \sup_{0\le t\le L,\ x\in K} \rho(f(t,x),g(t,x)), L^{-1}\} \tag{11}$$

and $\mathcal{D} := \{d_K : K \in C(X)\}$ a family of pseudo-metrics which generates the compact-open topology on $C(\mathbb{T} \times X, X)$, where $C(X)$ is the family of all compact subsets from X.

Remark 8. Note that for all $\varepsilon > 0$ the inequality $d_K(f, g) < \varepsilon$ is equivalent to $\sup\limits_{0 \le t \le \varepsilon^{-1},\ x \in K} \rho(f(t, x), g(t, x)) < \varepsilon$ (see, for example,[13, Ch.I] or [14, Ch.IV]).

Definition 9. Recall [2, Ch.I] that the collection $[x_1, x_2, \dots, x_k := y; t_1, t_2, \dots, t_{k-1}]$ is called a generalized chain joining x and y if the following conditions are fulfilled:

1. $t_i \ge t$;
2. $\rho(x, x_1) < \varepsilon$;
3. $\rho(\pi(t_i, x_i), x_{i+1}) < \varepsilon$ $(1 = 1, \dots, k-1)$.

Remark 9. In the book [2, Ch.I] it is shown that in the definition of chain recurrence the (ε, t, f)-chains can be replaced by generalized (ε, t, f)-chains.

Theorem 6. *Suppose that the following conditions hold:*

1. $\mathcal{M} \subset C(\mathbb{T} \times X, X)$ *is a compact subset from* $C(\mathbb{T} \times X, X)$*;*
2. *for all* $\pi \in \mathcal{M}$ *the dynamical system* $(X, \mathbb{T}, \pi)$ *is compact dissipative and* J_π *is its Levinson center;*
3. *the set* $J := \overline{\bigcup\{J_\pi : \pi \in \mathcal{M}\}}$ *is compact.*

Then the mapping $F : \mathcal{M} \mapsto 2^J$ *defined by equality* $F(\pi) := \mathfrak{R}(\pi)$ *is upper semi-continuous, where by* 2^J *the space of all compact subsets from* J *equipped with the Hausdorff metric is denoted.*

Proof. Let $\pi_n, \pi \in \mathcal{M}$ and $d_J(\pi_n, \pi) \to 0$, $a_n \in \mathfrak{R}(\pi_n)$ and $a_n \to a$ as $n \to \infty$. We need to show that $a \in \mathfrak{R}(\pi)$. Let ε be an arbitrary positive number and $0 < \delta < \varepsilon/4$. There exists a number $n_0 \in \mathbb{N}$ such that $\rho(a_n, a) < \delta$ and $d_J(\pi_n, \pi) < \delta$ for all $n \ge n_0$. Since $a_n \in \mathfrak{R}(\pi_n)$, then there is a $(\delta, \varepsilon^{-1}, \pi_n)$-chain from a_n to a_n, i.e., there exists a collection $\{x_0 = a_n, x_1, \dots, x_{k-1}, x_k = a_n; t_0, \dots, t_{k-1}\}$ such that

$$\rho(\pi_n(t_i, x_i), x_{i+1}) < \delta,\ \varepsilon^{-1} \le t_i \le 2\varepsilon^{-1}\ (i = 0, 1, \dots, k-1).$$

Thus the collection $[x_0, x_1, \dots, x_{k-1}, a; t_0, t_1, \dots, t_{k-1}]$ is a generalized $(2\delta, \varepsilon^{-1}, \pi_n)$-chain joining a with a. From the inequality $d_J(\pi_n, \pi) < \delta$ it follows that

$$\rho(\pi_n(t, x), \pi(t, x)) < \delta\ (x \in J,\ 0 \le t \le \delta^{-1} < 4\varepsilon^{-1})$$

and, consequently, the above indicated generalized $(2\delta, \varepsilon^{-1}, \pi_n)$-chain is also a generalized $(\varepsilon, \varepsilon^{-1}, \pi)$ chain from a to a. Since ε is an arbitrary positive number, then $a \in \mathfrak{R}(\pi)$. □

Lemma 12. *Suppose that $(X,\mathbb{T},\pi)$ is compact dissipative and J if its Levinson center, then $\omega_x \subseteq \mathfrak{R}(J) = \mathfrak{R}(\pi)$ for all $x \in X$.*

Proof. Let $x \in X$ be an arbitrary point. Since $(X,\mathbb{T},\pi)$ is compact dissipative, then ω_x is a nonempty, compact, and invariant subset of J, then $\mathfrak{R}(\omega_x) \subseteq \mathfrak{R}(J) = \mathfrak{R}(\pi)$. By Lemma 11 we have $\omega_x = \mathfrak{R}(\omega_x)$ and, consequently, $\omega_x \subseteq \mathfrak{R}(\pi)$. □

Lemma 13 (see [4, Ch.IV])**.** *If the compact invariant set Σ from X contains only a finite number of minimal sets, then the relation $\sim$ decomposes the set $\mathfrak{R}(\Sigma)$ into the finite number of different classes of equivalence (internally chain transitive sets).*

Remark 10. 1. Lemma 13 was established in [4, Ch.IV] for the two-sided (group) dynamical systems.

2. For the one-sided (semi-group) dynamical systems this statement may be proved by slight modifications of the arguments from [4, Ch.IV].

3. For two-sided dynamical systems ($\mathbb{T} = \mathbb{S}$) with infinite number of compact minimal subsets Lemma 13 remains true if in addition the dynamical system $(X,\mathbb{S},\pi)$ satisfies some condition of hyperbolicity (see Theorem 4.1 [4, Ch.IV]).

Lemma 14 (see [9])**.** *Let M be an isolated (local maximal) invariant set and $\mathfrak{R}$ be a compact internally chain transitive set for $(X,\mathbb{T},\pi)$. Assume that $M \bigcap \mathfrak{R} \neq \emptyset$ and $M \subseteq \mathfrak{R}$.*

Then

a. there exists a point $u \in \mathfrak{R} \setminus M$ such that $\omega_u \subseteq M$;

b. there exists a point $w \in \mathfrak{R} \setminus M$ and an entire trajectory $\gamma \in \Phi_w$ such that $\alpha_\gamma \subseteq M$.

Theorem 7. *Assume that the following conditions hold:*

1. the dynamical system $(X,\mathbb{T},\pi)$ is compactly dissipative and J is its Levinson center;

2. there exists a finite number n of compact minimal subsets $M_i \subseteq J$ $(i = 1,2,\dots,k)$ of $(X,\mathbb{T},\pi)$;

3. the collection of subsets $\{M_1, M_2, \dots, n\}$ does not admit k-cycles;

4. for all $x \in X$ there exists a number $i \in \{1,2,\dots,n\}$ such that $\omega_x = M_i$.

Then any compact internally chain transitive set $\mathfrak{R}_\lambda(\pi)$ is a minimal set of $(X,\mathbb{T},\pi)$, i. e., there exists $i \in \{1,2,\dots,n\}$ such that $\mathfrak{R}_\lambda = M_i$.

Proof. Let $\mathfrak{R}_\lambda(\pi)$ be a compact internally chain transitive set for $(X,\mathbb{T},\pi)$. Since $\mathfrak{R}_\lambda(\pi)$ is a compact positively invariant set, then by Birkhoff's theorem in $\mathfrak{R}_\lambda(\pi)$ there exists a nonempty compact minimal set $M_i \subseteq \mathfrak{R}_\lambda(\pi)$ $(i_1 \in \{1,2,\dots,n\})$. We will show that $\mathfrak{R}_\lambda(\pi) = M_{i_1}$. If we suppose that it is not true, then by Lemma 14 there exists a point $x_1 \in \mathfrak{R}_\lambda(\pi) \setminus M_{i_1}$ and an entire trajectory $\gamma_1 \in \Phi_{x_1}$ such that

$\alpha_{\gamma_1} \subseteq M_{i_1}$. By conditions of Theorem there exists a number $i_2 \in \{1, 2, \dots, n\}$ such that $\omega_{x_1} = M_{i_2}$. Since $M_{i_2} \subseteq \mathfrak{R}_\lambda(\pi)$ and $\mathfrak{R}_\lambda(\pi) \neq M_{i_2}$ then by Lemma 14 there exists a point $x_2 \in \mathfrak{R}_\lambda(\pi) \setminus M_{i_2}$ and an entire trajectory $\gamma_2 \in \Phi_{x_2}$ such that $\alpha_{\gamma_2} = M_{i_2}$ and there exists a number $i_3 \in \{1, 2, \dots, n\}$ such that $\omega_{x_2} = M_{i_3}$. Since there is only a finite number of M_i's, we will eventually arrive at a cyclic chain of some minimal sets of $(X, \mathbb{T}, \pi)$, which contradicts our assumption. □

Corollary 4. *Under the conditions of Theorem 7 we have* $\mathfrak{R}(\pi) = \coprod_{i=1}^n M_i$.

Theorem 8. *Suppose that* $(X, \mathbb{T}, \pi)$ *is a bounded dissipative dynamical system and* J *is its Levinson center. Then for every* $\delta > 0$ *and* $B \in \mathcal{B}(X)$ *there exists* $L = L(\delta, B) > 0$ *(*$L \in \mathbb{T}$*) such that*

$$\pi([0, L], x) \bigcap B(\mathfrak{R}(J), \delta) \neq \emptyset \text{ for all } x \in B,$$

i. e., for all $x \in B$ *there exists* $l = l(x) \in [0, L]$ *such that*

$$\pi(l, x) \in B(\mathfrak{R}(J), \delta).$$

Proof. If we suppose that the statement of Theorem is not true, then there are $\delta_0 > 0$, $B_0 \in \mathcal{B}(X)$, $L_n \geq n$ and $x_n \in B_0$ such that

$$\rho(\pi(t, x_n), \mathfrak{R}(J)) \geq \delta_0 \tag{12}$$

for all $t \in [0, L_n]$. Let $s_n := [L_n/2]$ and $\tilde{x}_n := \pi(s_n, x_n)$. Note that

$$\rho(\tilde{x}_n, J) = \rho(\pi(s_n, x_n), J) \leq \beta(\pi(s_n, B_0), J) \to 0 \tag{13}$$

as $n \to \infty$, because $s_n \to \infty$ and J attracts the bounded subset B_0 as $t \to +\infty$. From (13) it follows that the sequence $\{\tilde{x}_n\}$ is relatively compact. Thus, without loss of generality we can suppose that the sequence $\{\tilde{x}_n\}$ is convergent. Denote $\tilde{x} = \lim_{n\to\infty} \tilde{x}_n$, then by (13) we obtain $\tilde{x} \in J$. On the other hand by (12) we obtain

$$\rho(\pi(t, \tilde{x}_n), \mathfrak{R}(J)) = \rho(\pi(t + s_n, x_n), \mathfrak{R}(J)) \geq \delta_0 \tag{14}$$

for all $t \in [-s_n, s_n]$. Let $\gamma \in \mathcal{F}_{\tilde{x}}$ be the full trajectory of dynamical system $(X, \mathbb{T}, \pi)$ passing through $\{x\}$ at the initial moment $t = 0$ and defined by equality $\gamma(t) = \lim_{n\to\infty} \pi(t + s_n, x_n)$ for all $t \in \mathbb{S}$. Note that $\gamma(\mathbb{S}) \subseteq J$ because for every $t \in \mathbb{S}$ we have

$$\rho(\pi(t + s_n, x_n), J) \leq \rho(\pi(t + s_n, B_0), J) \tag{15}$$

for sufficiently large n, and passing to limit in (15) as $n \to \infty$ we obtain $\gamma(t) \in J$ for all $t \in \mathbb{S}$. By Lemma 12 we have $\omega_{\tilde{x}} \subseteq \mathfrak{R}(J)$. But from (14) it follows that $\gamma(t) \notin \mathfrak{R}(J)$ for all $t \in \mathbb{S}$ and, consequently, $\omega_{\tilde{x}} \bigcap \mathfrak{R}(J) = \emptyset$. The obtained contradiction proves our statement. Theorem is proved. □

Corollary 5. *Suppose that the following conditions hold:*

1. *$(X, \mathbb{T}, \pi)$ is a bounded dissipative dynamical system and J its Levinson center;*

2. *$(X, \mathbb{T}, \pi)$ is a gradient system;*

3. *$Fix(\pi) = \{p_1, p_2, \ldots, p_m\}$;*

4. *$Fix(\pi)$ does not contain any k-cycle ($k \geq 1$).*

Then for every $\delta > 0$ and $B \in \mathcal{B}(X)$ there exists $L = L(\delta, B) > 0$ ($L \in \mathbb{T}$) such that

$$\pi([0, L], B) \bigcap B(Fix(\pi), \delta) \neq \emptyset,$$

i. e., for all $x \in B$ there exists $l = l(x) \in [0, L]$ such that

$$\pi(l, x) \in B(Fix(\pi), \delta).$$

Proof. This statement follows from Theorems 7 and 8. □

Theorem 9. *Suppose that the following conditions are fulfilled:*

1. *the dynamical system $(X, \mathbb{T}, \pi)$ admits a compact global attractor J which attracts every bounded subset $B \in \mathcal{B}(X)$;*

2. *$\mathfrak{R}(J)$ consists of finite number of different classes of equivalence $\mathfrak{R}_1$ $\mathfrak{R}_2$, ..., $\mathfrak{R}_k$.*

Then for every $\tilde{\delta} > 0$ there exists $\delta \in (0, \tilde{\delta})$ such that for every $x \in B(\mathfrak{R}_i, \delta)$ ($i = \overline{1, k}$) with $\pi(t, x) \in B(\mathfrak{R}_i, \delta)$ for all $t \in [0, T)$ and $\pi(T, x) \notin B(\mathfrak{R}_i, \delta)$ we have $\pi(t, x) \notin B(\mathfrak{R}_i, \delta)$ for each $t \geq T$ (i. e., never returns again in $B(\mathfrak{R}_i, \delta)$ for all $t \geq T$).

Proof. By Lemma 4.3 [4, Ch.IV] in the collection $\{\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_k\}$ there is no r-cycles ($r \geq 1$). We will show that if we suppose that the statement of Theorem is not true, then we will have a contradiction this the fact formulated above. In fact. Suppose that Theorem is wrong, then there are $\mathfrak{R}_{i_0}$, $B(\mathfrak{R}_{i_0}, \delta_0)$ ($\delta_0 > 0$), $T_n \in \mathbb{T}$, $T'_n > T_n$ and a sequence $\{x_n\} \subset B(\mathfrak{R}_{i_0}, \delta_0)$ such that

$$\pi(T_n, x_n) \notin B(\mathfrak{R}_{i_0}, \delta_0) \text{ and } \pi(T'_n, x_n) \in B(\mathfrak{R}_{i_0}, 1/n).$$

Without loss of generality we can suppose that $\pi(t, x_n) \in B(\mathfrak{R}_{i_0}, \delta_0)$ for all $t \in [0, T_n)$.

Note that $T_n \to \infty$ as $n \to \infty$. If we suppose that it is not so, then we can consider that $\{T_n\}$ is bounded (otherwise we can extract a subsequence $\{T_{k_n}\}$ which converges to $+\infty$ as n goes to ∞), i. e., there exists a number $L > 0$ such that

$$\pi(t, x_n) \notin B(\mathfrak{R}_{i_0}, \delta_0) \tag{16}$$

for all $t \ge L$ and $n \in \mathbb{N}$. Since $x_n \in B(\mathfrak{R}_{i_0}, 1/n)$, then without loss of generality we can suppose that $\{x_n\}$ is convergent. Denote by $p := \lim_{n\to\infty} x_n$, then $p \in \mathfrak{R}_{i_0}$ and passing into limit in (16) as $n \to \infty$ we obtain

$$\pi(t,p) \notin B(\mathfrak{R}_{i_0}, \delta_0) \tag{17}$$

for all $t \ge L$. On the other hand

$$\pi(t,p) \in \mathfrak{R}_{i_0} \tag{18}$$

for all $t \ge 0$ because the set $\mathfrak{R}_{i_0}$ is invariant. Relations (17) and (18) are contradictory. The obtained contradiction proves our statement.

Denote by $\tilde{x}_n := \pi(T_n, x_n)$, then we have

1. $\tilde{x}_n \notin B(\mathfrak{R}_{i_0}, \delta_0)$ for all $n \in \mathbb{N}$;
2. $\pi(t, \tilde{x}_n) = \pi(t + T_n, x_n) \in B(\mathfrak{R}_{i_0}, \delta_0)$ for all $-T_n \le t < 0$;
3. $\pi(\tilde{T}'_n, \tilde{x}_n) \in B(\mathfrak{R}_{i_0}, 1/n)$ for all $n \in \mathbb{N}$, where $\tilde{T}'_n := T'_n - T_n > 0$.

Since $x_n \in B(\mathfrak{R}_{i_0}, 1/n)$, $T_n \to +\infty$ and $(X, \mathbb{T}, \pi)$ is compactly dissipative, then the sequence $\{\tilde{x}_n\}$ is relatively compact and without loss of generality we can suppose that it is convergent. Denote by $\tilde{x} := \lim_{n\to\infty} \tilde{x}_n$ and consider $\gamma \in \Phi_{\tilde{x}}$, where $\gamma(t) := \lim_{n\to\infty} \pi(t + T_n, x_n)$ for all $t \in \mathbb{S}$.

Note that $\tilde{T}'_n \to +\infty$ as $n \to \infty$. In fact, if we suppose that it is not true, then without loss of generality we can consider that $\{\tilde{T}'_n\}$ is bounded, for example, $\tilde{T}'_n \in [0, L]$ for all $n \in \mathbb{N}$, where L is some positive number. Let $l := \lim_{n\to\infty} \tilde{T}'_n$, then $l \in [0, L]$ (if it is necessary we can extract a convergent subsequence from $\{\tilde{T}'_n\}$). Then from (iii) we obtain $\pi(l, \tilde{x}) \in \mathfrak{R}_{i_0}$ and, consequently, $\tilde{x} \in \mathfrak{R}_{i_0}$ because $\mathfrak{R}_{i_0}$ is invariant. The obtained contradiction proves our statement.

We will show that $\gamma(t) \in J$ for all $t \in \mathbb{S}$. In fact

$$\rho(\pi(t + T_n, x_n), J) \le \beta(\pi(t + T_n, K), J) \to 0$$

as $n \to \infty$, where $K := \overline{\{x_n\}}$ and by bar the closure in the space X is denoted. Now we note that $\gamma(t) \in B(\mathfrak{R}_{i_0}, \delta_0)$ for all $t < 0$. Since the set $\mathfrak{R}_{i_0}$ is local maximal, then without loss of generality we can suppose that in $B(\mathfrak{R}_{i_0}, \delta_0)$ the invariant set $\mathfrak{R}_{i_0}$ is maximal and, consequently, $\alpha_\gamma \subseteq \mathfrak{R}_{i_0}$. On the other hand $\omega_{\tilde{x}} \subseteq \mathfrak{R}(J)$ and, consequently, there exists a number $i_1 \in \{1, 2, \dots, k\}$ such that $\omega_{\tilde{x}} \subseteq \mathfrak{R}_{i_1}$. Since the collection $\{\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_k\}$ has not r-cycles $(r \ge 1)$, then $i_1 \ne i_0$.

Since $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ and $\omega_{\tilde{x}} \subseteq \mathfrak{R}_{i_1}$, then by integral continuity for all $n \in \mathbb{N}$ there exists a number $T^1_n > 0$ such that $\pi(T^1_n, \tilde{x}_n) \in B(\mathfrak{R}_{i_1}, 1/n)$. Taking into account that $\tilde{T}'_n \to +\infty$ as $n \to \infty$ and Theorem 8 we can consider that $T^1_n \le \tilde{T}'_n$. On the other hand by Theorem 8 for all $n \in \mathbb{N}$ there exists $T^2_n \in (T^1_n, \tilde{T}'_n)$ such that $\pi(T^2_n, \tilde{x}_n) \notin B(\mathfrak{R}_{i_1}, \delta_0)$. Repeating the reasoning above for the set $\mathfrak{R}_{i_1}$ and the sequence $\{\tilde{x}_n\}$ we can find a full trajectory γ_1 so that $\alpha_{\gamma_1} \subseteq \mathfrak{R}_{i_1}$ and $\omega_{\tilde{x}_1} \subseteq \mathfrak{R}_{i_2}$, where $i_2 \neq i_0, i_1$ and $\tilde{x}_1 := \gamma_1(0)$.

Reasoning analogously we will construct a sequence $\{\gamma, \gamma_1, \dots, \gamma_p\}$ $(p \leq k-1)$ so that $\alpha_{\gamma_p} \subseteq \mathfrak{R}_{i_p}$ and $\omega_{\tilde{x}_p} \subseteq \mathfrak{R}_{i_{p+1}}$ $(\gamma_0 := \gamma)$. Since the family $\{\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_k\}$ contains a finite number of sets $\mathfrak{R}_p$, then after the finite number $q \leq k$ of steps we will have $\mathfrak{R}_{i_p} = R_{i_0}$, i. e., we will obtain a q-cycle. The obtained contradiction proves our Theorem. □

Corollary 6. *Suppose that the following conditions hold:*

1. $(X, \mathbb{T}, \pi)$ *is a bounded dissipative dynamical system and* J *its Levinson center;*
2. $(X, \mathbb{T}, \pi)$ *is a gradient system;*
3. $Fix(\pi) = \{p_1, p_2, \dots, p_m\}$*;*
4. $Fix(\pi)$ *does not contain any* k*-cycle* $(k \geq 1)$*.*

Then for every $\tilde{\delta} > 0$ *there exists* $\delta \in (0, \tilde{\delta})$ *such that for every* $x \in B(\mathfrak{R}_i, \delta)$ $(i = \overline{1,k})$ *with* $\pi(t,x) \in B(\mathfrak{R}_i, \delta)$ *for all* $t \in [0,T)$ *and* $\pi(T,x) \notin B(\mathfrak{R}_i, \delta)$ *we have* $\pi(t,x) \notin B(\mathfrak{R}_i, \delta)$ *for each* $t \geq T$ *(i. e., never returns again in* $B(\mathfrak{R}_i, \delta)$ *for all* $t \geq T$*).*

Proof. This statement follows from Theorems 8 and 9. □

Acknowledgements. Author was partially supported by grant FP7-PEOPLE-2012-IRSES-316338.

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Received January 24, 2014