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Invariant Characteristics of Special Compositions in Weyl Spaces W_N

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Abstract. In the present paper invariant characteristics of geodesic, chebyshevian and quasi-chebyshevian compositions $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$ in Weyl spaces $W_N(n_1 + n_2 + \cdots + n_p = N)$ are found with the help of the prolonged covariant differentiation. The characteristics of the spaces W_N which contain such special compositions are found.

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1 Preliminary

1. A prolonged covariant differentiation in W_N.

Let $W_N(g_{\alpha\beta}, T_\sigma)$ be Weyl space with a fundamental tensor $g_{\alpha\beta}$ and a complementary covector T_σ. Let us accept that the fundamental tensor $g_{\alpha\beta}$ is normed by the law (see [1], p.152)

$$\breve{g}_{\alpha\beta} = \lambda^2 g_{\alpha\beta} \ , \tag{1}$$

where λ is a function of the point. It is known (see [1], p.153) that after renormalization (1): the complementary covector T_σ transforms by the law $\breve{T}_\sigma = T_\sigma + \partial_\sigma ln\lambda$, which means T_σ is a normalizer; the reciprocal tensor $g^{\alpha\beta}$ to $g_{\alpha\beta}$ transforms by the law $g^{\alpha\beta} = \lambda^{-2} g^{\alpha\beta}$. The coefficients of the connectedness $\Gamma^\sigma_{\alpha\beta}$ of the Weyl space W_N have the presentation $\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} g^{\sigma\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) - (T_\alpha \delta^\sigma_\beta + T_\beta \delta^\sigma_\alpha - T_\nu g^{\nu\sigma} g_{\alpha\beta})$ (see [1], p.154).

Let N independent fields of directions $\underset{\sigma}{v}^\alpha$ $(\sigma, \alpha = 1, 2, \ldots, N)$ be given in W_N. Renorm the fields of directions $\underset{\sigma}{v}^\alpha$ by the condition [8]

$$g_{\alpha\beta} \underset{\sigma}{v}^\alpha \underset{\sigma}{v}^\beta = 1. \tag{2}$$

The reciprocal covectors $\overset{\sigma}{v}_\alpha$ are defined by the following equalities

$$\underset{\sigma}{v}^\alpha \ \overset{\sigma}{v}_\beta = \delta^\alpha_\beta \Longleftrightarrow \underset{\beta}{v}^\sigma \ \overset{\alpha}{v}_\sigma = \delta^\alpha_\beta. \tag{3}$$

The renormalization of the fundumental tensor accompanies with the following renorming $\underset{\sigma}{\breve{v}}^\alpha = \lambda^{-1} \underset{\sigma}{v}^\alpha$, $\ \breve{\overset{\sigma}{v}}\alpha = \lambda \overset{\sigma}{v}_\alpha$.

According to (see [1], p.152) the fundamental tensor $g_{\alpha\beta}$ and the complementary covector T_σ satisfy the equalities

$$\nabla_\sigma\ g_{\alpha\beta} = 2T_\sigma\ g_{\alpha\beta}\ ,\ \nabla_\sigma\ g^{\alpha\beta} = -2T_\sigma\ g^{\alpha\beta} \tag{4}$$

According to [7] the pseudo-quantities $A \in W_N$ which after renormalization of the fundamental tensor $g_{\alpha\beta}$ by the formula (1) transform by the law $\breve{A} = \lambda^k A$ are called satellites of $g_{\alpha\beta}$ with a weight $\{k\}$. Hence $g^{\alpha\beta}\{-2\}$, $\underset{\sigma}{v}^{\alpha}\{-1\}$ $\overset{\sigma}{v}_{\alpha}\{1\}$.

The existence of the normalizer T_σ allows to introduce a prolonged covariant differentiation of the satellites $A\ \{k\}$ of the tensor $g_{\alpha\beta}$ by the formula $\overset{\circ}{\nabla}_\sigma\ A = \nabla_\sigma A - kT_\sigma A$ [8]. According to [8,9] we have.

$$\overset{\circ}{\nabla}_\sigma\ g_{\alpha\beta} = 0\ ,\ \overset{\circ}{\nabla}_\sigma\ g^{\alpha\beta} = 0\ ,\ \overset{\circ}{\nabla}_\sigma\ \underset{\alpha}{v}^{\beta} = \nabla_\sigma\ \underset{\alpha}{v}^{\beta} + T_\sigma \underset{\alpha}{v}^{\beta},\ \overset{\circ}{\nabla}_\sigma\ \overset{\alpha}{v}_{\beta} = \nabla_\sigma\ \overset{\alpha}{v}_{\beta} - T_\sigma \overset{\alpha}{v}_{\beta}. \tag{5}$$

Ozdeger obtained significant results in the understanding the geometry of Weyl and Einstein-Weyl manifolds [11], using the prolonged covariant differentiation, introduced in [8].

2. Compositions in $\boldsymbol{W_N}$.

Consider in the space W_N the composition $X_m \times X_{N-m}$ of two base manifolds X_m and X_{N-m}, i.e. their topological product. Two positions $P(X_m)$ and $P(X_{N-m})$ of these base manifolds pass through any point of the space $W_N(X_m \times X_{N-m})$ [2]. According to [2] and [3] any composition is completely defined with the field of the affinor a_α^β, satisfying the condition

$$a_\alpha^\sigma a_\sigma^\beta = \delta_\alpha^\beta. \tag{6}$$

According to [4] the projecting affinors $\overset{m}{a}\,{}_\alpha^\beta$, $\overset{N-m}{a}\,{}_\alpha^\beta$ are defined by the equalities $\overset{m}{a}\,{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta + a_\alpha^\beta)$, $\overset{N-m}{a}\,{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta - a_\alpha^\beta)$. For an arbitrary vector v^α we have $v^\alpha = \overset{m}{a}\,{}_\sigma^\alpha\ v^\sigma + \overset{N-m}{a}\,{}_\sigma^\alpha\ v^\sigma = \underset{m}{V}^\alpha + \underset{N-m}{V}{}^\alpha$, where $\underset{m}{V}^\alpha = \overset{m}{a}\,{}_\sigma^\alpha\ v^\sigma \in P(X_m)$, $\underset{N-m}{V}{}^\alpha = \overset{N-m}{a}\,{}_\sigma^\alpha\ v^\sigma \in P(X_{N-m})$. The partial projections or the full ones of an arbitrary tensor are defined analogously.

3. Derivative equations in $\boldsymbol{W_N}$.

For the independent fields of directions $\underset{\sigma}{v}^{\alpha}$ $(\sigma, \alpha = 1, 2, \ldots, N)$ and their reciprocal covectors $\overset{\sigma}{v}_{\alpha}$, defined by (3), are fulfilled the following derivative equations [8,9]

$$\overset{\circ}{\nabla}_\sigma\ \underset{\alpha}{v}^{\beta} = \underset{\alpha}{\overset{\nu}{T}}{}_\sigma \underset{\nu}{v}^{\beta}\ ,\ \ \overset{\circ}{\nabla}_\sigma\ \overset{\alpha}{v}_{\beta} = -\underset{\nu}{\overset{\alpha}{T}}{}_\sigma \overset{\nu}{v}_{\beta}\ , \tag{7}$$

where $\underset{\alpha}{\overset{\beta}{T}}{}_\beta\ \{0\}$. We obtain, using the integrability condition of (7), the next equality $\nabla_{[\alpha} \underset{\sigma}{\overset{\sigma}{T}}{}_{\beta]} + \underset{\nu}{\overset{\sigma}{T}}{}_{[\beta} \underset{\sigma}{\overset{\nu}{T}}{}_{\alpha]}$ =0 [8]. Let us denote by $(\underset{\beta}{v})$ the lines, defined from the field of directions $\underset{\beta}{v}^{\alpha}$ and by $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ the net, defined from the independent fields of directions $\underset{\sigma}{v}^{\alpha}$, $(\sigma = 1,2,\dots,N)$. It is known that the field of directions $\underset{\sigma}{v}^{\alpha}$ is parallelly translated along the lines $(\underset{\beta}{v})$ if and only if $\nabla_\nu\, \underset{\sigma}{v}^{\alpha}\underset{\beta}{v}^{\nu} = \mu\underset{\sigma}{v}^{\alpha}$, where μ is an arbitrary function of the point. According to (5) the last equality can be written in the form

$$\overset{\circ}{\nabla}_\nu\, \underset{\sigma}{v}^{\alpha}\underset{\beta}{v}^{\nu} = \mu\underset{\sigma}{v}^{\alpha}. \tag{8}$$

2 Coordinate net in W_N

Let us chose the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ as a coordinate one. From (2) and $g_{\alpha\beta}\underset{\sigma}{v}^{\alpha}\underset{\nu}{v}^{\beta} = cos\underset{\sigma\nu}{\omega}$ it follows that in the parameters of the coordinate net

$$\begin{array}{c} g_{\alpha\beta} = \underset{\alpha}{f}\underset{\beta}{f}cos\underset{\alpha\beta}{\omega}, \\ \underset{1}{v}^{\alpha}(\frac{1}{\underset{1}{f}},0,0,\dots,0), \quad \underset{2}{v}^{\alpha}(0,\frac{1}{\underset{2}{f}},0,\dots,0), \quad \dots, \quad \underset{N}{v}^{\alpha}(0,0,0,\dots,\frac{1}{\underset{N}{f}}), \\ \overset{1}{v}_{\alpha}(\underset{1}{f},0,0,\dots,0), \quad \overset{2}{v}_{\alpha}(0,\underset{2}{f},0,\dots,0), \quad \dots, \quad \overset{N}{v}_{\alpha}(0,0,0,\dots,\underset{N}{f}), \end{array} \tag{9}$$

where $\underset{\alpha}{f} = \underset{\alpha}{f}(\overset{\sigma}{u}),\ \underset{\alpha}{f}\{1\},\ \underset{\alpha\beta}{\omega} = \underset{\alpha\beta}{\omega}(\overset{\sigma}{u})\ ,\ \underset{\alpha\beta}{\omega}\{0\},\ \sigma = 1,2,\dots,N.$

Lemma 1. *When the net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ *is chosen as a coordinate one then there exist the following relations between the coefficients* $\underset{\alpha}{\overset{\beta}{T}}_{\sigma}$ *from the derivative equations (7) and the coefficients of the connection* $\Gamma^{\sigma}_{\alpha\beta}$

$$\underset{\alpha}{\overset{\beta}{T}}_{\sigma} = \frac{\underset{\beta}{f}}{\underset{\alpha}{f}}\Gamma^{\beta}_{\sigma\alpha}\ , \quad \alpha \neq \beta\ ; \quad \underset{\alpha}{\overset{\alpha}{T}}_{\sigma} = \Gamma^{\alpha}_{\sigma\alpha} - \partial_\sigma ln(\underset{1}{f}\underset{2}{f}\dots\underset{N}{f}) + NT_\sigma\ . \tag{10}$$

Proof. Using (3), (5) and (7) we obtain

$$\underset{\alpha}{\overset{\beta}{T}}_{\sigma} = \partial_\sigma\underset{\alpha}{v}^{\nu}\overset{\beta}{v}_{\nu} + \Gamma^{\tau}_{\sigma\nu}\underset{\alpha}{v}^{\nu}\overset{\beta}{v}_{\tau} + T_\sigma\delta^{\beta}_{\alpha}\ . \tag{11}$$

After applying (9) in (11) we establish the validity of (10). □

3 Weyl spaces of compositions $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$

Let us introduce the notations:

$$\begin{aligned}
&\alpha, \beta, \gamma, \delta, \sigma, \nu, \tau = 1, 2, \ldots, N; \ i_1, j_1, k_1, s_1 = 1, 2, \ldots, n_1; \\
&\bar{i}_1, \bar{j}_1, \bar{k}_1, \bar{s}_1 = n_1 + 1, n_1 + 2, \ldots, N; \\
&i_2, j_2, k_2, s_2 = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2; \\
&\bar{i}_2, \bar{j}_2, \bar{k}_2, \bar{s}_2 = 1, 2, \ldots, n_1, n_1 + n_2 + 1, n_1 + n_2 + 2, \ldots, N; \\
&i_3, j_3, k_3, s_3 = n_1 + n_2 + 1, n_1 + n_2 + 2, \ldots, n_1 + n_2 + n_3; \\
&\bar{i}_3, \bar{j}_3, \bar{k}_3, \bar{s}_3 = 1, 2, \ldots, n_1 + n_2 + n_3 + 1, n_1 + n_2 + n_3 + 2, \ldots, N; \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&i_p, j_p, k_p, s_p = n_1 + n_2 + \cdots + n_{p-1} + 1, \\
&n_1 + n_1 + n_2 + \cdots + n_{p-1} + 2, \ldots, N; \\
&\bar{i}_p, \bar{j}_p, \bar{k}_p, \bar{s}_p = 1, 2, \ldots, n_1 + n_2 + \cdots + n_{p-1}\ .
\end{aligned} \tag{12}$$

Following [10] we shall consider the affinors

$$\overset{n_m}{a}{}_{\alpha}^{\beta} = \underset{i_m}{v}{}^{\beta} \overset{i_m}{v}{}_{\alpha} - \underset{\bar{i}_m}{v}{}^{\beta} \overset{\bar{i}_m}{v}{}_{\alpha} \quad \textit{for any} \quad m = 1, 2, \ldots, p. \tag{13}$$

The affinors (13) have weight $\{0\}$. According to (3) the affinors (13) satisfy (6), i.e. they define the following compositions $X_{n_1} \times X_{N-n_1}, X_{n_2} \times X_{N-n_2}, \ldots, X_{n_p} \times X_{N-n_p}$. Let us consider the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$ and let us denote the positions of the manifolds $X_{n_1}, X_{n_2}, \ldots, X_{n_p}$, by $P(X_{n_1})$, $P(X_{n_2})$, $\ldots$, $P(X_{n_p})$, respectively.

The affinors

$$\overset{m}{a}{}_{\alpha}^{\beta} = \underset{i_m}{v}{}^{\beta} \overset{i_m}{v}{}_{\alpha}\ , \quad m = 1, 2, \ldots, p, \tag{14}$$

with weight $\{0\}$ will be called the projective affinors of the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$.

From (3) and (14) follow $\overset{1}{a}{}_{\alpha}^{\beta} + \overset{2}{a}{}_{\alpha}^{\beta} + \cdots + \overset{p}{a}{}_{\alpha}^{\beta} = \delta_{\alpha}^{\beta}$, $\overset{m}{a}{}_{\alpha}^{\beta} \overset{m}{a}{}_{\sigma}^{\alpha} = \overset{m}{a}{}_{\sigma}^{\beta}$, $\overset{m}{a}{}_{\alpha}^{\beta} \overset{l}{a}{}_{\sigma}^{\alpha} = 0$, where $m, l = 1, 2, \ldots, p, \quad m \neq l$. If v^{β} is an arbitrary vector, then $v^{\beta} = \overset{1}{a}{}_{\alpha}^{\beta} v^{\alpha} + \overset{2}{a}{}_{\alpha}^{\beta} v^{\alpha} + \cdots + \overset{p}{a}{}_{\alpha}^{\beta} v^{\alpha} = \underset{1}{V}{}^{\beta} + \underset{2}{V}{}^{\beta} + \cdots + \underset{p}{V}{}^{\beta}$, where $\underset{1}{V}{}^{\beta} = \overset{1}{a}{}_{\alpha}^{\beta} v^{\alpha} \in P(X_{n_1})$, $\underset{2}{V}{}^{\beta} = \overset{2}{a}{}_{\alpha}^{\beta} v^{\alpha} \in P(X_{n_2})$, $\ldots$, $\underset{p}{V}{}^{\beta} = \overset{p}{a}{}_{\alpha}^{\beta} v^{\alpha} \in P(X_{n_p})$.

With the help of the projective affinors (14) the fundamental tensor $g_{\alpha\beta}$ can be presented in the form $g_{\alpha\beta} = \overset{1}{G}_{\alpha\beta} + \overset{2}{G}_{\alpha\beta} + \cdots + \overset{p}{G}_{\alpha\beta} + 2\overset{12}{G}_{\alpha\beta} + 2\overset{13}{G}_{\alpha\beta} + \cdots + 2\overset{p-1p}{G}_{\alpha\beta}$, where $\overset{m}{G}_{\alpha\beta} = \overset{m}{a}{}_{\alpha}^{\sigma} \overset{m}{a}{}_{\beta}^{\nu} g_{\sigma\nu}$, $\overset{ml}{G}_{\alpha\beta} = \overset{m}{a}{}_{(\alpha}^{\sigma} \overset{l}{a}{}_{\beta)}^{\nu} g_{\sigma\nu}$ and $m, l = 1, 2, \ldots, p, \ m \neq l$.

The tensors $\overset{m}{G}_{\alpha\beta}$ are full projections of the fundamental tensor $g_{\alpha\beta}$ on the positions $P(X_{n_m})$ and they define metrics on these positions. Following [5] the tensors $\overset{m}{G}_{\alpha\beta}$ will be called positional fundamental tensors. They satisfy the equalities $\overset{m}{a}{}^{\sigma}_{\alpha}\overset{m}{G}_{\sigma\beta} = \overset{m}{a}{}^{\sigma}_{\beta}\overset{m}{G}_{\alpha\sigma} = \overset{m}{G}_{\alpha\beta}$, $\overset{m}{a}{}^{\sigma}_{\alpha}\overset{l}{G}_{\sigma\beta} = \overset{m}{a}{}^{\sigma}_{\beta}\overset{l}{G}_{\alpha\sigma} = 0$, when $m \neq l$. Following [5] the tensors $\overset{ml}{G}_{\alpha\beta}$ will be called hybridian tensors. They satisfy the equalities $\overset{m}{a}{}^{\sigma}_{\alpha}\overset{l}{a}{}^{\nu}_{\beta}\overset{ml}{G}_{\sigma\nu} = \frac{1}{2}\overset{m}{a}{}^{\sigma}_{\alpha}\overset{l}{a}{}^{\nu}_{\beta}\, g_{\sigma\nu}$, $\overset{m}{a}{}^{\sigma}_{\alpha}\overset{m}{a}{}^{\nu}_{\beta}\overset{ml}{G}_{\sigma\nu} = 0$.

4 Special compositions $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$ in W_N

Definition 1. The composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ will be called geodesic if for any $m = 1, 2, \dots, p$ the position $P(X_{n_m})$ is parallelly translated along any line of the manifold X_{n_m}.

Theorem 1. *The composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is geodesic if and only if the coefficients from the derivative equations (7) satisfy the equalities*

$$\underset{i_m}{\overset{\bar{k}_m}{T}}{}_{\sigma}\, \underset{s_m}{v}{}^{\sigma} = 0\ ,\ \ for\ \ any\ \ m = 1, 2, \dots, p. \tag{15}$$

Proof. According to (8) the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$ is geodesic if and only if $\overset{\circ}{\nabla}_{\sigma} \underset{i_m}{v}{}^{\alpha} \underset{s_m}{v}{}^{\sigma} = \mu \underset{i_m}{v}{}^{\alpha}$ for any $m = 1, 2, \dots, p$. From (7) and the last equality we obtain $\underset{i_m}{\overset{\nu}{T}}{}_{\sigma}\ \underset{\nu}{v}{}^{\alpha}\ \underset{s_m}{v}{}^{\sigma} = \mu \underset{i_m}{v}{}^{\alpha}$. Now after contraction by $\overset{\tau}{v}_{\alpha}$ we find $\underset{i_m}{\overset{\tau}{T}}{}_{\sigma}\ \underset{s_m}{v}{}^{\sigma} = \mu\ \delta^{\tau}_{i_m}$, from where (15) follows. □

From (9), (10) and Theorem 1 follows the validity of the following statement:

Corollary 1. *If the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is geodesic then:*

i) In the parameters of the coordinate net the coefficients of the derivative equations (7) satisfy the equalities $\underset{i_m}{\overset{\bar{k}_m}{T}}{}_{s_m} = 0$ for any $m = 1, 2, \dots, p$;

ii) In the parameters of the coordinate net the coefficients of the connection satisfy the equalities $\Gamma^{\bar{k}_m}_{s_m i_m} = 0$ for any $m = 1, 2, \dots, p$.

If the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is geodesic and the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ is chosen as a coordinate one, then using Corollary 1, for the components of the tensor of the curvature $R_{\alpha\beta\gamma}{}^{\delta}$ we obtain $R_{i_m j_m k_m}{}^{\bar{s}_m} = 0$ for any $m = 1, 2, \dots, p$.

Definition 2. The composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ will be called chebyshevian if for any $m, l = 1, 2, \dots, p$ and $m \neq l$, the position $P(X_{n_m})$ is parallelly translated along any line of the manifold X_{n_l}.

Theorem 2. *The composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is chebyshevian if and only if the coefficients from the derivative equations (7) satisfy the equalities*

$$\underset{i_m}{\overset{\overline{k}_m}{T}}{}_{\sigma} \underset{s_l}{v}{}^{\sigma} = 0 \ , \ \textit{for} \quad \textit{any} \quad m, l = 1, 2, \ldots, p, m \neq l. \tag{16}$$

Proof. According to (8) the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$ is chebyshevian if and only if $\overset{\circ}{\nabla}_{\sigma} \underset{i_m}{v}{}^{\alpha} \underset{s_l}{v}{}^{\sigma} = \mu \underset{i_m}{v}{}^{\alpha}$ for any $m = 1, 2, \ldots, p$. From (7) and the last equality we obtain (16). □

From (9), (10) and Theorem 2 follows the validity of the following statement:

Corollary 2. *If the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is chebyshevian then:*

i) In the parameters of the coordinate net the coefficients of the derivative equations (7) satisfy the equalities $\underset{i_m}{\overset{\overline{k}_m}{T}}{}_{s_l} = 0$ for any $m, l = 1, 2, \ldots, p, \quad m \neq l$;

ii) In the parameters of the coordinate net the coefficients of the connection satisfy the equalities $\Gamma^{\overline{k}_m}_{s_l \ i_m} = 0$ for any $m, l = 1, 2, \ldots, p, \quad m \neq l$.

Theorem 3. *If the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is chebyshevian then the space W_N is Riemannian and the metric tensor has in the chosen coordinate system the presentation*

$$g_{i_l i_m} = \underset{i_l}{f}(\overset{i_l}{u}) \ \underset{i_m}{f}(\overset{i_m}{u}) \ cos \ \underset{i_l i_m}{\omega}(\overset{i_l}{u}, \overset{i_m}{u}) \ . \tag{17}$$

Proof. Let the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ be chebyshevian. We chose the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{N}{v})$ as a coordinate one. Then from (4) and Corollary 2 we obtain

$$\partial_{i_m} g_{i_l i_r} = 2T_{i_m} g_{i_l i_r}, \text{ for any } m, l, r = 1, 2, \ldots, p, \ m \neq l, \ m \neq r. \tag{18}$$

From (18) it follows $T_\sigma = grad$, i.e. W_n is Riemannian. Let us renormalize the fundumental tensor $g_{\alpha\beta}$ such that $T_\sigma = 0$, (see [1], p.157). Then the equalities (18) accept the form $\partial_{i_m} g_{i_l i_r} = 0$, from where (17) follows. □

Let now the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ be chebyshevian and X_{n_m} are one-dimensional manifolds. Then the composition defines a chebyshevian net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{N}{v})$. According to Theorem 3 W_N is Riemannian. Using (17) and changing the variables, we obtain for the metric tensor of the Riemannian space $g_{\alpha\beta} = cos \ \underset{\alpha\beta}{\omega}(\overset{\alpha}{u}, \overset{\beta}{u})$.

Let us consider an orthogonal composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$, which means that at any point of the space any two directions $\underset{m}{V}{}^{\alpha} \in P(X_{n_m})$ and $\underset{l}{V}{}^{\alpha} \in P(X_{n_l})$, when $m, l = 1, 2, \ldots, p, \ m \neq l$, are orthogonal. In this case $g_{\alpha\beta} \underset{m}{V}{}^{\alpha} \underset{l}{V}{}^{\beta} = 0$. Since $\underset{m}{V}{}^{\alpha} = \overset{m}{a}{}^{\alpha}_{\sigma} v^{\sigma}$, $\underset{l}{V}{}^{\alpha} = \overset{l}{a}{}^{\alpha}_{\sigma} v^{\sigma}$, then $g_{\alpha\beta} \underset{m}{V}{}^{\alpha} \underset{l}{V}{}^{\beta} = 0 \Longleftrightarrow g_{\alpha\beta} \overset{m}{a}{}^{\alpha}_{\sigma} \overset{l}{a}{}^{\beta}_{\nu} v^{\sigma} u^{\nu} =$

$g_{\alpha\beta}\overset{l}{a}{}_{\sigma}^{\alpha}\overset{m}{a}{}_{\nu}^{\beta}v^{\sigma}u^{\nu}=0$. Because v^{α} and u^{α} are arbitrary vector fields, then $g_{\alpha\beta}\overset{m}{a}{}_{\sigma}^{\alpha}\overset{l}{a}{}_{\nu}^{\beta}=g_{\alpha\beta}\overset{l}{a}{}_{\sigma}^{\alpha}\overset{m}{a}{}_{\nu}^{\beta}=0$, from where it follows $\overset{ml}{G}_{\alpha\beta}=0$. Hence $g_{\alpha\beta}=\overset{1}{G}_{\alpha\beta}+\overset{2}{G}_{\alpha\beta}+\cdots+\overset{p}{G}_{\alpha\beta}$.

Theorem 4. *The orthogonal composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ is chebyshevian if and only if it is geodesic one.*

Proof. Let the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ be orthogonal. Then from $\underset{i_m}{v}{}^{\alpha}\in P(X_{n_m})$, $\underset{i_k}{v}{}^{\alpha}\in P(X_{n_k})$ it follows $g_{\alpha\beta}\underset{i_m}{v}{}^{\alpha}\underset{i_k}{v}{}^{\beta}=0$ for any $m,k=1,2,\ldots,p,\ m\neq k$. After prolonged covariant differentiation of the last equality and taking into account (5) and (7) we find $g_{\alpha\beta}\overset{j_k}{\underset{i_m}{T}}{}_{\sigma}\underset{j_k}{v}{}^{\alpha}\underset{i_k}{v}{}^{\beta}+g_{\alpha\beta}\overset{j_m}{\underset{i_k}{T}}{}_{\sigma}\underset{i_m}{v}{}^{\alpha}\underset{j_m}{v}{}^{\beta}=0$. Now after contraction by $\underset{s_k}{v}{}^{\sigma}$ we obtain

$$g_{\alpha\beta}\overset{j_k}{\underset{i_m}{T}}{}_{\sigma}\underset{s_k}{v}{}^{\sigma}\underset{j_k}{v}{}^{\alpha}\underset{i_k}{v}{}^{\beta}+g_{\alpha\beta}\overset{j_m}{\underset{i_k}{T}}{}_{\sigma}\underset{s_k}{v}{}^{\sigma}\underset{i_m}{v}{}^{\alpha}\underset{j_m}{v}{}^{\beta}=0. \tag{19}$$

From (19), Theorem 1 and Theorem 2 the validity of the Theorem 4 follows.

The compositions $X_m\times X_{N-m}$ for which the positions $P(X_m)$ and $P(X_{N-m})$ are quasi-parallelly translated along any line of the manifold X_{N-m} and X_m, respectively are studied in [2, 5, 6].

Let us consider the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$. According to [2,5,6] and (7) the positions $P(X_{n_m})$ will be quasi-parallelly translated along any line of the manifold X_{n_k} if and only if

$$\overset{\circ}{\nabla}_{\sigma}\ \underset{i_m}{v}{}^{\alpha}\underset{j_k}{v}{}^{\sigma}=\lambda_{i_m}\underset{j_k}{v}{}^{\alpha}+\overset{s_m}{\underset{i_m}{T}}{}_{\sigma}\underset{s_m}{v}{}^{\alpha}\underset{j_k}{v}{}^{\sigma},\quad m\neq k. \tag{20}$$

The vector λ_{i_m} has the weight $\{-1\}$. □

Definition 3. The composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ will be called quasi-chebyshevian if for any $m,k=1,2,\ldots,p,\ m\neq k$, the positions $P(X_{n_m})$ are quasi-parallelly translated along any line of the manifold X_{n_k} .

Theorem 5. *The composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ is quasi-chebyshevian if and only if the coefficients from the derivative equations (7) satisfy the equalities*

$$\overset{\overline{s}_m}{\underset{i_m}{T}}{}_{\sigma}\ \underset{j_k}{v}{}^{\sigma}=\lambda_{i_m}\delta_{j_k}^{\overline{s}_m},\ \ for\ \ any\ \ m,k=1,2,\ldots,p,\ m\neq k. \tag{21}$$

Proof. According to (7) and (20) the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ will be quasi-chebyshevian if and only if $\overset{\overline{s}_m}{\underset{i_m}{T}}{}_{\sigma}\ \underset{\overline{s}_m}{v}{}^{\alpha}\underset{j_k}{v}{}^{\sigma}=\lambda_{i_m}\underset{j_k}{v}{}^{\alpha}$. The last equalities are equivalent to (21). □

From (9), (10) and Theorem 5 follows the validity of the following statement:

Corollary 3. *If the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is quasi-chebyshevian then:*

i) In the parameters of the coordinate net the coefficients of the derivative equations (7) satisfy the equalities $\frac{1}{\underset{j_k}{f}} \overset{\overline{s}_m}{\underset{i_m}{T}}{}_{j_k} = \lambda_{i_m} \delta^{\overline{s}_m}_{j_k},\ for\ \ any\ \ m, k = 1, 2, \dots, p,\ m \neq k.$

ii) In the parameters of the coordinate net the coefficients of the connection satisfy the equalities $\Gamma^{\overline{s}_m}_{j_k\ i_m} = \psi_{i_m} \delta^{\overline{s}_m}_{j_k}$ *for any* $m, k = 1, 2, \dots, p,\ \ m \neq k$, *where the vector* $\psi_{i_m} = \frac{\lambda_{i_m}}{f_{i_m}}$ *has the weight* $\{0\}$.

Following [2] the vector ψ_{i_m} will be called a vector of the quasi-parallel translation. If for any $m, k = 1, 2, \dots, p$ $\psi_{i_m} = 0$, then according to Theorem 2 the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ will be chebyshevian.

Theorem 6. *The composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is geodesic or chebyshevian, or quasi-chebyshevian if and only if the projecting affinors (14) satisfy for any $m, k = 1, 2, \dots, p,\ \ m \neq k$ the equalities*

$$\begin{aligned} &\overset{m}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu} = 0, \\ &\overset{k}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu} = 0, \\ &\overset{k}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu} - \psi_{\sigma} \overset{m}{a}{}^{\sigma}_{\delta} \overset{k}{a}{}^{\beta}_{\alpha} = 0, \end{aligned} \tag{22}$$

respectively.

Proof. Let the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ be chosen as a coordinate one. In the parameters of this coordinate net we have $\overset{m}{a}{}^{\beta}_{\alpha} = \delta^{i_m}_{s_m}$, $\overset{k}{a}{}^{\beta}_{\alpha} = \delta^{i_k}_{s_k}$. For the components of the tensors $\overset{m}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu}$, $\overset{k}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu}$, $\overset{k}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu} - \psi_{\sigma} \overset{m}{a}{}^{\sigma}_{\delta} \overset{k}{a}{}^{\beta}_{\alpha}$, which are diffrent from zero, we find

$$\begin{aligned} &\overset{m}{a}{}^{\sigma}_{i_m} \overset{m}{a}{}^{\nu}_{j_m} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\overline{s}_m}_{\nu} = \Gamma^{\overline{s}_m}_{i_m j_m}, \\ &\overset{k}{a}{}^{\sigma}_{i_m} \overset{m}{a}{}^{\nu}_{j_k} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\overline{s}_m}_{\nu} = \Gamma^{\overline{s}_m}_{i_m j_k}, \\ &\overset{k}{a}{}^{\sigma}_{i_k} \overset{m}{a}{}^{\nu}_{j_m} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\overline{s}_m}_{\nu} - \psi_{\sigma} \overset{m}{a}{}^{\sigma}_{j_m} \overset{k}{a}{}^{\overline{s}_m}_{l_k} = \psi_{j_m} \delta^{\overline{s}_m}_{i_k}. \end{aligned} \tag{23}$$

From Corollaries 1, 2, 3 and (23) follows (22). □

References

[1] Norden A. P. *Affinely Connected Space.* Monographs, Moscow, 1976 (in Russian).

[2] Norden A., Timofeev G. *Invariant Tests of the Special Compositions in Multivariate Spaces* Izv. Vyssh. Uchebn. Zaved., Mat., 1972, **8**, 81–89 (in Russian).

[3] Yano K. *Affine Connexions in an Almost Product Space* Kodai Math. Sem. Rep., 1959, **11(1)**, 1–24.

[4] Walker A. *Connexions for Parallel Distributions in the Large, II.* Q. J. Math., 1958, **9(35)**, 221–231.

[5] Timofeev G. N. *Invariant Tests of the Special Compositions in Weyl Spaces.* Izv. Vyssh. Uchebn. Zaved., Mat., 1976, **1**, 87–99 (in Russian).

[6] Leontiev E. K. *Classification of the Special Bundles and Compositions in Many-dimensional Spaces.* Izv. Vyssh. Uchebn. Zaved. Mat., 1967, **5(60)**, 40–51 (in Russian).

[7] Zlatanov G., Norden A. P. *Orthogonal Trajectories of Geodesic Fields.* Izv. Vyssh. Uchebn. Zaved. Mat., 1975, **7(158)**, 42–46 (in Russian).

[8] Zlatanov G. *Nets in an n–demensional Space of Weyl.* C. R. Acad. Bulgare Sci., 1988, **41(10)**, 29–32 (in Russian).

[9] Zlatanov G. *Special Networks in the n–dimensional Space of Weyl W_n.* Tensor (N. S.), 1989, **48**, 234–240.

[10] Zlatanov G. *Compositions, Generated by Special Nets in Affinely Connected Spaces.* Serdica Math. J., 2002, **28**, 189–200.

[11] Ozdeger A. *Conformal and Generalized Concircular Mappings of Einstein-Weyl Manifolds.* Acta Math. Sci. Ser. B, Engl. Ed., 2010, **30(5)**, 1739–1745.

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