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One new class of cubic systems with maximum number of invariant lines omitted in the classification of J. Llibre and N. Vulpe

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Abstract. We present a new class of cubic systems with invariant lines of total multiplicity 9, including the line at infinity endowed with its own multiplicity. This class is different from the 23 classes included in the classification given in [4] by J. Llibre and N. Vulpe.

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Consider real cubic systems, i. e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(a,x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(a,x,y) \end{aligned} \tag{1}$$

with real coefficients and variables x and y. The polynomials p_i and q_i $(i = 0, 1, 2, 3)$ are homogeneous polynomials of degree i in x and y.

Let

$$\mathbf{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

be the polynomial vector field corresponding to systems (1).

A straight line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ satisfies

$$\mathbf{X}(f) = uP(x,y) + vQ(x,y) = (ux + vy + w)R(x,y)$$

for some polynomial $R(x,y)$ if and only if it is *invariant* under the flow of the systems. If some of the coefficients u, v, w of an invariant straight line belongs to $\mathbb{C}\setminus\mathbb{R}$, then we say that *the straight line is complex*; otherwise *the straight line is real.* Note that, since systems (1) are real, if a system has a complex invariant straight line $ux + vy + w = 0$, then it also has its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$.

Definition 1. We say that an invariant affine straight line $f(x,y) = ux+vy+w = 0$ (respectively the line at infinity $Z = 0$) for a cubic vector field $\mathbf{X}$ has multiplicity m if there exists a sequence of real cubic vector fields $\mathbf{X}_k$ converging to $\mathbf{X}$, such

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that each $\mathbf{X}_k$ has m (respectively $m-1$) distinct invariant affine straight lines $f_i^j = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0,0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbb{C}^3$, converging to $f = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$ (respectively m).

Definition 2 (see [5])**.** Consider a planar cubic system (1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

According to [1] the maximum number of invariant straight lines taking into account their multiplicities (including the line at infinity) for cubic systems is 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities have been made in [4]. In this paper the authors have detected 23 classes of such cubic systems and have constructed the corresponding canonical forms. Moreover they proved that modulo affine group and time rescaling each such class is represented by a specific cubic system without parameters.

We are interested in the classification of cubic systems which possess invariant lines of total multiplicity 8. For this purpose we split the whole family of cubic systems in four subfamilies, depending on the number of distinct infinite singularities.

It is well known that the infinite singularities (real or complex) of systems (1) are determined by the linear factors of the polynomial

$$C_3 = yp_3(x,y) - xq_3(x,y).$$

In the paper [2] the cubic systems with 4 distinct infinite singularities are examined and it was proved that there exist 17 classes of such cubic systems with invariant lines of total multiplicity 8. All possible distinct configurations of invariant straight lines are determined and the corresponding necessary and sufficient conditions are constructed in terms of affine invariant polynomials (see for instance [6, 7]).

The second family of cubic systems which possess 3 distinct infinite singularities has been examined in [3]. We proved that this family of systems can only have 5 distinct configurations of invariant lines of total multiplicity 8 and determined the corresponding affine invariant criteria.

Now the family of cubic systems with two distinct infinite singularities is under examination. And in the case when infinite singularities of cubic systems are determined by one simple and one triple factors of C_3, we have detected a new class of cubic systems with maximum number (nine) of invariant straight lines. In what follows we show that this class is omitted in the classification given by J. Llibre and N. Vulpe in [4].

Indeed, considering the family of cubic systems with 8 invariant lines (including the line at infinity and including multiplicities) which in addition possesses two infinite singularities, we found out that a subfamily of these systems could be brought via affine transformation and time rescaling to the canonical form

$$\dot{x} = x(r + 2x + x^2), \quad \dot{y} = y(r + 2x), \quad 0 \neq r \in \mathbb{R}, \tag{2}$$

depending on one parameter. We observe that these systems possess invariant lines of total multiplicity 8. More precisely, we have the invariant affine lines: $x = 0$ (triple), $y = 0$, $x^2 + 2x + r = 0$ (simple real or complex or real double) and the line at infinity ($Z = 0$), which is double.

We detected that in the case $r = 8/9$ the obtained system

$$\dot{x} = x(2 + 3x)(4 + 3x)/9, \quad \dot{y} = 2(4 + 9x)y/9 \tag{3}$$

possesses invariant lines of total multiplicity 9, and namely: $x = 0$ (triple), $x = -2/3$ (double), $x = -4/3$ and $y = 0$ (both simple) and the line at infinity ($Z = 0$) (double).

To prove this it is sufficient to present the following corresponding perturbed systems

$$\dot{x} = x(2 + 3x)(4 + 3x)/9, \quad \dot{y} = 2y(1 + \varepsilon y)(4 + 9x - 4\varepsilon y)/9,$$

which possess the following 8 invariant affine lines:

$$x = 0, \quad y = 0, \quad x = -2/3, \quad x = -4/3, \quad 3x - 4\varepsilon y = 0,$$
$$3x - 2\varepsilon y = 0, \; 1 + \varepsilon y = 0, \; 3x - 2\varepsilon y + 2 = 0$$

Thus system (3) indeed possesses invariant lines of total multiplicity 9 (including the infinite one).

On the other hand in [4] nine classes of cubic systems with two infinite singularities determined by one simple and one triple factors of C_3 are given and their corresponding configurations are presented in Figures 14–22.

Considering the configuration of invariant lines of system (3) given in Fig. 1 we observe that this configuration is different from configurations given in Figures 14–22 [4].

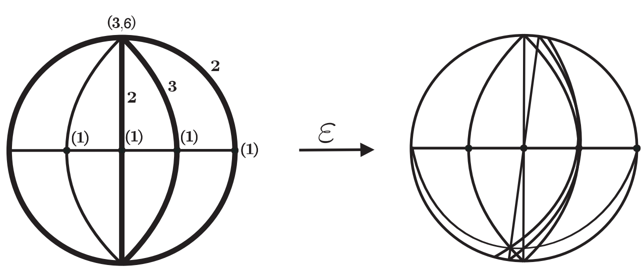

Figure 1. The configuration of invariant lines corresponding to system (3)

Remark. In the above configuration if an invariant straight line has multiplicity $k > 1$, then the number k appears near the corresponding straight line and this line is in bold face. We indicate next to the real singular points of the system, located on the invariant straight lines, their corresponding multiplicities. In the case of infinite singularities we denote by '(a, b)' the maximum number a (respectively b) of infinte (respectively finite) singularities which can be obtained by perturbation of the multiple point.

References

[1] Artes J., Grünbaum D., Llibre J. *On the number of invariant straight lines for polynomial differential systems*. Pacific Journal of Mathematics, 1998, **184**, 317–327.

[2] Bujac C., Vulpe N. *Cubic systems with invariant lines of total multiplicity eight and with four distinct infinite singularities*. Preprint No. 10, 2013, Universitat Autónoma de Barcelona, 1–51.

[3] Bujac C., Vulpe N. *Cubic systems with invariant lines of total multiplicity eight and with three distinct infinite singularities*. CRM Preprint No. 3331, Montreal, December 2013, 1–30.

[4] Llibre J., Vulpe N. I. *Planar cubic polynomial differential systems with the maximum number of invariant straight lines*. Rocky Mountain J. Math., 2006, **38**, 1301–1373.

[5] Schlomiuk D., Vulpe N. *Global classification of the planar Lotka–Volterra differential systems according to their configurations of invariant straight lines*. Journal of Fixed Point Theory and Applications, 2010, **8**, No. 1, 69 p.

[6] Sibirskii K. S. *Introduction to the algebraic theory of invariants of differential equations*. Translated from the Russian. Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988.

[7] Vulpe N. I. *Polynomial bases of comitants of differential systems and their applications in qualitative theory*. Kishinev, "Shtiintsa", 1986 (in Russian).

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