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# Semilattices of $r$-archimedean subdimonoids

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**Abstract.** We characterize dimonoids which are semilattices of $r$-archimedean ($\ell$-archimedean, $(t;r)$-archimedean) subdimonoids.

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## 1 Introduction

Dimonoids were introduced by J.-L. Loday [1] for the study of properties of Leibniz algebras. Dialgebras, which are based on the notion of a dimonoid, have been studied by many mathematicians (see, for example, [1-4]). It is well-known that the notion of a dimonoid generalizes the notion of a digroup [5]. Digroups play a prominent role in an important open problem from the theory of Leibniz algebras. Dimonoids were studied in the papers of the author (see, for example, [6–11]). Moreover, note that algebras with two associative operations (so-called bisemigroups) were considered earlier in some other aspects in the paper of B. M. Schein [12]. The study of connections between dimonoids and bisemigroups was started in [11].

In this work we characterize dimonoids which are bands of subdimonoids. In Section 2 we give necessary definitions, auxiliary results (Lemma 1 and Theorem 3) and some properties of dimonoids (Lemma 2, Theorem 4 and Corollary 1). Putcha [13] gave necessary and sufficient conditions under which an arbitrary semigroup is a semilattice of $r$-archimedean ($\ell$-archimedean, $t$-archimedean) semigroups. In Section 3 we extend Putcha's results to the case of dimonoids (Theorem 5).

## 2 Preliminaries

A nonempty set $D$ equipped with two binary associative operations $\prec$ and $\succ$ satisfying the following axioms:

$$(x \prec y) \prec z = x \prec (y \succ z),$$

$$(x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \prec y) \succ z = x \succ (y \succ z)$$

for all $x, y, z \in D$, is called a dimonoid. If the operations of a dimonoid coincide, then the dimonoid becomes a semigroup.

Different examples of dimonoids can be found in [1, 6–11].

The notion of a diband of subdimonoids was introduced in [6] and investigated in [7]. Recall this definition.

A dimonoid $(D, \prec, \succ)$ is called an idempotent dimonoid or a diband if $x \prec x = x = x \succ x$ for all $x \in D$. If $\varphi : S \to T$ is a homomorphism of dimonoids, then the corresponding congruence on $S$ will be denoted by $\Delta_\varphi$.

Let $S$ be an arbitrary dimonoid, $J$ be some idempotent dimonoid. Let

$$\alpha : S \to J : x \mapsto x\alpha$$

be a homomorphism. Then every class of the congruence $\Delta_\alpha$ is a subdimonoid of the dimonoid $S$, and the dimonoid $S$ itself is a union of such dimonoids $S_\xi,\ \xi \in J$, that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \,|(x; t) \in \Delta_\alpha\},$$

$$S_\xi \prec S_\varepsilon \subseteq S_{\xi \prec \varepsilon}, \quad S_\xi \succ S_\varepsilon \subseteq S_{\xi \succ \varepsilon},$$

$$\xi \neq \varepsilon \Rightarrow S_\xi \bigcap S_\varepsilon = \varnothing.$$

In this case we say that $S$ is decomposable into a diband of subdimonoids (or $S$ is a diband $J$ of subdimonoids $S_\xi,\ \xi \in J$). If $J$ is a band (=idempotent semigroup), then we say that $S$ is a band $J$ of subdimonoids $S_\xi,\ \xi \in J$. If $J$ is a commutative band, then we say that $S$ is a semilattice $J$ of subdimonoids $S_\xi,\ \xi \in J$.

We denote the set of positive integers by $N$. Let $(D, \prec, \succ)$ be a dimonoid and $a \in D$, $n \in N$. Denote the power $n$ of an element $a$ with respect to the operation $\prec$ (respectively, $\succ$) by $a^n$ (respectively, by $n\,a$).

**Lemma 1** (see [8], Lemma 2.4)**.** *Let $(D, \prec, \succ)$ be an arbitrary dimonoid. For all $x \in D,\ n \in N$*

*(i) $x^n \succ x = (n+1)x$;*

*(ii) $x \prec nx = x^{n+1}$.*

A semigroup $S$ is called $r$-archimedean (respectively, $\ell$-archimedean) if for all $a, b \in S$ there exist $x \in S^1,\ n \in N$ such that $b^n = ax$ (respectively, $b^n = xa$). A semigroup $S$ is called $t$-archimedean if for all $a, b \in S$ there exist $x, y \in S^1,\ n \in N$ such that $b^n = ax = ya$.

Let $(D, \prec, \succ)$ be a dimonoid. We denote the semigroup $(D, \prec)$ (respectively, $(D, \succ)$) with an identity by $D^1_\prec$ (respectively, by $D^1_\succ$).

**Lemma 2.** *Let $(D, \prec, \succ)$ be an arbitrary dimonoid.*

*(i) If $(D, \prec)$ is an $r$-archimedean semigroup, then $(D, \succ)$ is an $r$-archimedean semigroup.*

*(ii) If $(D, \succ)$ is an $\ell$-archimedean semigroup, then $(D, \prec)$ is an $\ell$-archimedean semigroup.*

*(iii) If $(D, \prec)$ is a $t$-archimedean semigroup, then $(D, \succ)$ is an $r$-archimedean semigroup.*

*(iv) If $(D, \succ)$ is a $t$-archimedean semigroup, then $(D, \prec)$ is an $\ell$-archimedean semigroup.*

*Proof.* (i) Let $(D, \prec)$ be an $r$-archimedean semigroup. Then for all $a, b \in D$ there exist $x \in D^1_{\prec}$, $n \in N$ such that $a \prec x = b^n$. Multiply both parts of the last equality by $b$ with respect to the operation $\succ$:

$$(a \prec x) \succ b = a \succ (x \succ b) = b^n \succ b = (n+1)b$$

according to the axiom of a dimonoid and Lemma 1 (i). So, $(D, \succ)$ is an $r$-archimedean semigroup.

(ii) Let $(D, \succ)$ be an $\ell$-archimedean semigroup. Then for all $a, b \in D$ there exist $x \in D^1_{\succ}$, $n \in N$ such that $x \succ a = nb$. Multiply both parts of the last equality by $b$ with respect to the operation $\prec$:

$$b \prec (x \succ a) = (b \prec x) \prec a = b \prec nb = b^{n+1}$$

according to the axiom of a dimonoid and Lemma 1 (ii). So, $(D, \prec)$ is an $\ell$-archimedean semigroup.

The proofs of (iii) and (iv) are similar. □

A semigroup $S$ is called archimedean if for all $a, b \in S$ there exist $x, y \in S^1$, $n \in N$ such that $b^n = xay$. A dimonoid is called archimedean if its both semigroups are archimedean.

Let $(D, \prec, \succ)$ be a dimonoid, $a, b \in D$. Introduce the following notations: $a_{\prec}|b$ if $b \in D^1_{\prec} \prec a \prec D^1_{\prec}$ and $a_{\succ}|b$ if $b \in D^1_{\succ} \succ a \succ D^1_{\succ}$.

**Theorem 3** (see [8], Theorem 4.1). *A dimonoid $(D, \prec, \succ)$ is a semilattice of archimedean subdimonoids if and only if for all $a, b \in D$,*

$$a_{\prec}|b \Rightarrow a^2_{\prec}|b^n \quad \textit{for some} \quad n \in N. \tag{1}$$

Dually, the following theorem can be proved.

**Theorem 4.** *A dimonoid $(D, \prec, \succ)$ is a semilattice of archimedean subdimonoids if and only if for all $a, b \in D$,*

$$a_{\succ}|b \Rightarrow 2a_{\succ}|nb \quad \textit{for some} \quad n \in N. \tag{2}$$

From Theorem 4 we obtain

**Corollary 1.** *Let $(D, \prec, \succ)$ be a dimonoid. Then*

*(i) $(D, \prec, \succ)$ with a medial semigroup $(D, \succ)$ is a semilattice of archimedean subdimonoids;*

*(ii) $(D, \prec, \succ)$ with a commutative operation $\succ$ is a semilattice of archimedean subdimonoids;*

*(iii) $(D, \prec, \succ)$ with an exponential semigroup $(D, \succ)$ is a semilattice of archimedean subdimonoids;*

*(iv) $(D, \prec, \succ)$ with a weakly exponential semigroup $(D, \succ)$ is a semilattice of archimedean subdimonoids.*

Dually to Corollary 4.1 from [8], this corollary can be proved.

## 3 The main result

Observe that a commutative dimonoid was decomposed into a semilattice of archimedean subdimonoids in [6]. In [9] a free commutative dimonoid was constructed and this dimonoid was decomposed into a semilattice of archimedean subdimonoids. In [8] we gave necessary and sufficient conditions under which an arbitrary dimonoid is a semilattice of archimedean subdimonoids (see also Theorems 3 and 4).

In this section we give necessary and sufficient conditions under which an arbitrary dimonoid is a semilattice of $r$-archimedean ($\ell$-archimedean, $(t;r)$-archimedean) subdimonoids.

Let $(D, \prec, \succ)$ be a dimonoid and $a, b \in D$. Introduce the following notations: $a_{\prec}|_r b$ if $a \prec x = b$ for some $x \in D^1_{\prec}$; $a_{\prec}|_\ell b$ if $x \prec a = b$ for some $x \in D^1_{\prec}$; $a_{\succ}|_\ell b$ if $x \succ a = b$ for some $x \in D^1_{\succ}$; $a_{\prec}|_t b$ if $a_{\prec}|_r b$ and $a_{\prec}|_\ell b$.

A dimonoid will be called $r$-archimedean (respectively, $\ell$-archimedean) if both its semigroups are $r$-archimedean (respectively, $\ell$-archimedean). A dimonoid $(D, \prec, \succ)$ will be called $(t;r)$-archimedean if $(D, \prec)$ is a $t$-archimedean semigroup and $(D, \succ)$ is an $r$-archimedean semigroup.

**Theorem 5.** *Let $(D, \prec, \succ)$ be an arbitrary dimonoid. Then*

*(i) $(D, \prec, \succ)$ is a semilattice of $r$-archimedean subdimonoids if and only if for all $a, b \in D$,*

$$a_{\prec}|b \Rightarrow a_{\prec}|_r b^n \quad \textit{for some} \quad n \in N. \tag{3}$$

*(ii) $(D, \prec, \succ)$ is a semilattice of $\ell$-archimedean subdimonoids if and only if for all $a, b \in D$,*

$$a_{\succ}|b \Rightarrow a_{\succ}|_\ell nb \quad \textit{for some} \quad n \in N. \tag{4}$$

*(iii) $(D, \prec, \succ)$ is a semilattice of $(t;r)$-archimedean subdimonoids if and only if for all $a, b \in D$,*

$$a_{\prec}|b \Rightarrow a_{\prec}|_t b^n \quad \textit{for some} \quad n \in N. \tag{5}$$

*Proof.* (i) Let the condition (3) hold. By Theorem 3 (1) from [13] the condition (1) follows from (3). Hence according to Theorem 3 $(D, \prec, \succ)$ is a semilattice $Y$ of archimedean subdimonoids $(D_i, \prec, \succ), i \in Y$. From Theorem 3 (1) [13] it follows that $(D_i, \prec)$, $i \in Y$, is an $r$-archimedean semigroup. Then by Lemma 2 (i) $(D_i, \succ)$, $i \in Y$, is an $r$-archimedean semigroup. Thus, $(D_i, \prec, \succ)$, $i \in Y$, is an $r$-archimedean subdimonoid of $(D, \prec, \succ)$.

The necessity follows from Theorem 3 (1) [13].

(ii) Let the condition (4) hold. By Theorem 3 (2) from [13] the condition (2) follows from (4). Hence according to Theorem 4 $(D, \prec, \succ)$ is a semilattice $Y$ of archimedean subdimonoids $(D_i, \prec, \succ)$, $i \in Y$. From Theorem 3 (2) [13] it follows that $(D_i, \succ)$, $i \in Y$, is an $\ell$-archimedean semigroup. Then by Lemma 2 (ii) $(D_i, \prec)$, $i \in Y$, is an $\ell$-archimedean semigroup. Thus, $(D_i, \prec, \succ)$, $i \in Y$, is an $\ell$-archimedean subdimonoid of $(D, \prec, \succ)$.

The necessity follows from Theorem 3 (2) [13].

(iii) Let the condition (5) hold. By Theorem 3 (3) from [13] the condition (1) follows from (5). Hence according to Theorem 3 $(D, \prec, \succ)$ is a semilattice $Y$ of archimedean subdimonoids $(D_i, \prec, \succ)$, $i \in Y$. From Theorem 3 (3) [13] it follows that $(D_i, \prec)$, $i \in Y$, is a $t$-archimedean semigroup. Then by Lemma 2 (iii) $(D_i, \succ)$, $i \in Y$, is an $r$-archimedean semigroup. Thus, $(D_i, \prec, \succ)$, $i \in Y$, is a $(t; r)$-archimedean subdimonoid of $(D, \prec, \succ)$.

The necessity follows from Theorem 3 (3) [13]. □

Theorem 5 extends Theorem 3 from [13] about necessary and sufficient conditions under which an arbitrary semigroup is a semilattice of $r$-archimedean ($\ell$-archimedean, $t$-archimedean) semigroups.

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