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The Variational Approach to Nonlinear Evolution Equations

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Abstract. In this paper, we present a few recent existence results via variational approach for the Cauchy problem

$$\frac{dy}{dt}\,(t) + A(t)y(t) \ni f(t), \quad y(0) = y_0, \quad t \in [0, T],$$

where $A(t) : V \to V'$ is a nonlinear maximal monotone operator of subgradient type in a dual pair (V, V') of reflexive Banach spaces. In this case, the above Cauchy problem reduces to a convex optimization problem via Brezis–Ekeland device and this fact has some relevant implications in existence theory of infinite-dimensional stochastic differential equations.

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1 Introduction

Consider the Cauchy problem

$$\begin{aligned} &\frac{dy}{dt}\,(t) + A(t)y(t) \ni f(t), \quad t \in (0, T), \\ &y(0) = y_0, \end{aligned} \tag{1.1}$$

where $y : [0, T] \to V$, $\frac{dy}{dt} : (0, T) \to V'$, $f : (0, T) \to V'$ and $y_0 \in H$. Here V is a real reflexive Banach space with the dual V' and H is a real Hilbert space such that $V \subset H \subset V'$ algebraically and topologically.

The scalar product on H and the duality pairing between V and V' are both denoted by $(\cdot, \cdot)$ and the latter coincides with the scalar product of H on $H \times H \subset V \times V'$. Here $A(t) : V \to V'$, $t \in (0, T)$, is a family of maximal monotone operators on $V \times V'$ of the form (see, e.g.,[4])

$$A(t) = \partial\varphi(t, \cdot) \quad \text{a.e. } t \in (0, T), \tag{1.2}$$

where $\varphi(t, \cdot) : V \to \overline{\mathbb{R}}^* =]-\infty, +\infty]$ is a family of convex and lower-semicontinuous functions and $\partial\varphi(t, \cdot) : V \to V'$ is the subdifferential of $\varphi(t, \cdot)$ (see, e.g., [4, 5]).

By strong solution to (1.1) we mean a measurable function $y : (0, T) \to V$ which is H-valued continuous and V'-absolutely continuous on $[0, T]$ and satisfies a.e. equation (1.1) on $(0, T)$ along with the initial value condition $y(0) = y_0 \in H$.

We recall that

$$\partial\varphi(t,y)=\{z\in V';\ \varphi(t,y)\le\varphi(t,u)+(y-u,z),\ \forall u\in V\},\ y\in V, \tag{1.3}$$

and the conjugate function $\varphi^*(t,\cdot):V'\to\overline{\mathbb{R}}^*$ is defined by (see, e.g., [4,5])

$$\varphi^*(t,z)=\sup\{(y,z)-\varphi(t,y);\ y\in V\},\ \forall z\in V'. \tag{1.4}$$

We recall also the duality relations (see [5])

$$\varphi(t,y)+\varphi^*(t,z)\ \ge\ (y,z),\quad \forall y\in V,\ z\in V', \tag{1.5}$$

$$\varphi(t,y)+\varphi^*(t,z)\ =\ (y,z)\quad \text{iff } z\in\partial\varphi(t,y). \tag{1.6}$$

By virtue of (1.5) and (1.6), we may rewrite equation (1.1) as

$$\frac{dy}{dt}(t)+z(t)=f(t),\quad \varphi(t,y(t))+\varphi^*(t,z(t))=(y(t),z(t)),t\in(0,T).$$

Equivalently,

$$\varphi(t,y(t))+\varphi^*\left(t,f(t)-\frac{dy}{dt}(t)\right)=\left(y(t),f(t)-\frac{dy}{dt}(t)\right),\qquad \text{a.e. } t\in(0,T). \tag{1.7}$$

In other words, any strong solution y to (1.1) can be viewed as solution to the minimization problem

$$\operatorname{Min}\left\{\int_0^T(\varphi(t,y(t))+\varphi^*\left(t,f(t)-\frac{dy}{dt}(t)\right)-\left(y(t),f(t)-\frac{dy}{dt}(t)\right)dt\right\}. \tag{1.8}$$

The exact formulation of (1.8) will be given later on, but is easily seen that if one takes the minimum in (1.8) on the space

$$\mathcal{W}_p=\left\{y\in L^p(0,T;V),\ \frac{dy}{dt}\in L^{p'}(0,T;V');\ \frac{1}{p}+\frac{1}{p'}=1,\ 1<p<\infty\right\},$$

then (1.8) reduces to the convex optimization problem

$$\operatorname{Min}\left\{\int_0^T\left(\varphi(t,y(t))+\varphi^*(t,f(t)-\frac{dy}{dt}(t)-(y(t),f(t))\right)dt \right.$$
$$\left. +\frac{1}{2}(|y(T)|^2-|y_0)|^2);\ y\in\mathcal{W}_p\right\}. \tag{1.9}$$

Conversely, one might expect that every solution y to problem (1.9) is a strong solution to the Cauchy problem (1.1) and we shall see that this is indeed the case under suitable assumptions on $A(t)=\partial\varphi(t,\cdot)$. This is the fundament idea behind the variational approach to equation (1.1), which goes back to the influential works [7,8] of Brezis and Ekeland (see, also, Nayrolles [12]). Now this is known as Brezis &

Ekeland principle and it was a fertile idea used later on in a variety of situations (see [1–3, 9–11, 13, 14]). Though, in general, the equivalence of problems (1.1) and (1.9) is still an open problem and it is not true in general, this principle leads to a variational formulation of a large class of nonlinear Cauchy problems which, from the point of view of mathematical physics and numerical computation, represents a great advantage. As a matter of fact, by this device the Cauchy problem (1.1) reduces to a convex optimization problem for which a large set of strategies which belong to convex analysis are applicable. This approach is, in particular, useful for the time-dependent Cauchy problems of the form (1.1) for which a complete existence theory is known only in a few situations requiring either time-regularity of the operator $A(t)$ or polynomial growth conditions from V to V' (see [4], Section 4.4). In applications to stochastic differential equations of the form

$$\begin{aligned} &dX + AX(t)dt = dW(t), \quad t \in (0,T),\\ &X(0) = y_0, \end{aligned} \tag{1.10}$$

in a probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, where $A = \partial\varphi$ and W is a Wiener process on H, we are lead to an equation of the form (1.1) by the transformation $y = X - W_t$ and get the random differential equation

$$\begin{aligned} &\frac{dy}{dt} + A(y + W(t)) = 0, \quad t \in (0,T),\\ &y(0) = y_0. \end{aligned} \tag{1.11}$$

Since the function $t \to W(t)$ is not smooth, we are lead to an equation of the form (1.1) when $A(t)y \equiv A(y + W(t))$ for which the standard existence theory is not applicable but which can be reformulated in terms of (1.9). This problem, which is discussed in details in [4, 6], represents a viable and promising approach to the existence theory of infinite-dimensional nonlinear stochastic differential equations.

Notation and definitions

If Y is a Banach space with the norm $\|\cdot\|_Y$, we denote by $L^p(0,T;Y)$, $1 \le p \le \infty$, the space of all Y-measurable functions $u : (0,T) \to Y$ with $\|u\|_Y \in L^p(0,T)$. By $C([0,T];Y)$ we denote the space of all continuous Y-valued functions on $[0,T]$ and by $W^{1,p}([0,T];Y)$ the Sobolev space $\big\{y \in L^p(0,T;Y);\, \frac{d}{dt} \in L^p(0,T);Y)\big\}$, where $\frac{dy}{dt}$ is taken in the sense of vectorial distributions on $(0,T)$. Equivalently, $W^{1,p}([0,T];Y)$ is the space of absolutely continuous functions $u : [0,T] \to Y$ which are a.e. differentiable and $\frac{d}{dt} \in L^p(0,T;Y)$. (See [4, 5].)

Everywhere in the following, $\mathcal{O}$ is an open and bounded subset of the Euclidean space $\mathbb{R}^d$, $d \ge 1$, with smooth boundary $\partial\mathcal{O}$ (of class C^2, for instance) and $W^{k,p}(\mathcal{O})$, $k \in \mathbb{N}$, $1 \le p \le \infty$, are standard Sobolev spaces on $\mathcal{O}$, i.e.,

$$W^{k,p}(\mathcal{O}) = \{u \in L^p(\mathcal{O});\ D^\alpha u \in L^p(\mathcal{O}),\ |\alpha| \le k\}. \tag{1.12}$$

$W_0^{k,p}(\mathcal{O})$ is the subset of functions in $W^{k,p}(\mathcal{O})$ which are of trace zero on $\partial\mathcal{O}$.

We set $H_0^1(\mathcal{O}) = W_0^{1,2}(\mathcal{O})$.

A multivalued function (graph) $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is said to be *maximal monotone* if it is monotone, that is, $(v_1 - v_2)(u_1 + u_2) \geq 0$ for $v_i \in \beta(u_i)$, $i = 1, 2$, and the range of $u \to u + \beta(u)$ is all of $\mathbb{R}$. Any maximal monotone graph $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is of the form $\beta = \partial j$, where $j : \mathbb{R} \to]-\infty, +\infty]$ is convex and lower-semicontinuous. (This is the potential function corresponding to β.)

2 Existence for the Cauchy problem (1.1)

2.1 The main results

We study here problem (1.1) under the following hypotheses:

(i) *V is a real Banach space with the dual V' and H is a real Hilbert space such that $V \subset H \subset V'$ algebraically and topologically.*

Denote by $(\cdot, \cdot)$ the pairing between V and V' and, respectively, the scalar product on H. The norms of V, V' and H are denoted by $\| \cdot \|_V$, $\| \cdot \|_{V'}$ and $| \cdot |_H$.

(ii) *$A(t)y = \partial\varphi(t, y)$ a.e. $t \in (0, T)$, $\forall y \in V$, where $\varphi : (0, T) \times V \to \mathbb{R}$ is measurable in t on $(0, T)$ and lower-semicontinuous on V with respect to y. There are $\alpha_1, \alpha_2 > 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $1 < p_1 \leq p_2 < \infty$ such that*
$$\gamma_1 + \alpha_2 \|u\|_V^{p_1} \leq \varphi(t, u) \leq \gamma_2 + \alpha_2 \|u\|_V^{p_2}, \ \forall u \in V, \text{ a.e. } t \in (0, T). \tag{2.1}$$

Instead of (ii) we consider the following alternative weaker assumption on φ.

(iii) *$A(t) = \partial\varphi(t, \cdot)$, where $\varphi : (0, T) \times H \to \mathbb{R}$ is measurable in t, convex and lower-semicontinuous in y on H and for each $M > 0$ there is $C_M > 0$ independent of t such that*

$$\varphi(t, u) \leq C_M \text{ a.e. } t \in (0, T), \ \|u\|_V \leq M, \tag{2.2}$$
$$\gamma_1 + \alpha_1 \|u\|_V^{p_1} \leq \varphi(t, u), \ \forall u \in V, \text{ a.e. } t \in (0, T). \tag{2.3}$$

It should be mentioned that both hypotheses (ii), (iii) imply that $\varphi(t, \cdot)$ is continuous on V for almost all $t \in (0, T)$ but no differentiability conditions so $A(t) = \partial\varphi(t, \cdot)$ might be multivalued as well.

Hypothesis (iv) below is a symmetry condition on $u \to \varphi(t, u)$ for large $\|u\|_V$.

(iv) *There are $C_1, C_2 \in R^+$ such that*
$$\varphi(t, -u) \leq C_1 \varphi(t, u) + C_2, \ \ \forall u \in V, \text{ a.e. } t \in (0, T). \tag{2.4}$$

Theorems 2.1, 2.2 below are the main results.

Theorem 2.1. *Under hypotheses* (i), (ii), (iv)*, for each $y_0 \in V$ and $f \in L^{p_1'}(0, T; V')$ there is a unique strong solution to* (1.1) *satisfying*

$$y \in L^{p_1}(0, T; V) \cap C([0, T]; H) \cap W^{1, p_2'}([0, T]; V'), \tag{2.5}$$

where $\frac{1}{p_i}+\frac{1}{p_i'}=1,\ 1<p_i<\infty, i=1,2.$

Moreover, y is the solution to the minimization problem

$$\begin{aligned}\text{Min}\Big\{ \int_0^T \Big(\varphi(t,u(t))+\varphi^*\Big(t,f(t)-\frac{du}{dt}(t)\Big)-(u(t),f(t))\Big)dt \\ +\frac{1}{2}|u(T)|_H^2;\ u\in L^{p_1}(0,T;V)\cap W^{1,p_2'}([0,T];V'),\ u(0)=y_0\Big\}.\end{aligned} \tag{2.6}$$

We have also

Theorem 2.2. *Under hypotheses* (i), (iii), (iv) *for each* $y_0\in V$ *and* $f\in L^{p_1'}(0,T;V')$ *there is a unique strong solution to* (1.1) *such that*

$$y^*\in L^{p_1}(0,T;V)\cap C([0,T];H)\cap W^{1,1}([0,T];V). \tag{2.7}$$

Moreover, y^* is the solution to the minimization problem (2.6)*.*

2.2 Examples to PDEs

Now, we pause briefly to see how Theorems 2.1 and 2.2 apply to a few standard parabolic nonlinear boundary value problems.

Example 2.1. (Semilinear parabolic equations) Consider the boundary value problem

$$\begin{aligned} &\frac{\partial y}{\partial t}-\Delta y+\beta(t,y)\ni f(t,\xi), && t\in(0,T),\xi\in\mathcal{O},\\ &y(0,\xi)=y_0(\xi), && \xi\in\mathcal{O},\\ &y(t,\xi)=0, && \text{on } (0,T)\times\partial\mathcal{O}.\end{aligned} \tag{2.8}$$

Here $\mathcal{O}\subset\mathbb{R}^d$, $d\geq 1$, is a bounded open domain with smooth boundary $\partial\mathcal{O}$ and $\beta:(0,T)\times\mathbb{R}\to 2^{\mathbb{R}}$ is a maximal monotone graph in y for almost all $t\in(0,T)$ and is measurable in t.

Denote by $j(t,\cdot):\mathbb{R}\to\mathbb{R}$ the potential associated with $\beta(t,\cdot)$ that is,

$$\partial_r j(t,r)=\beta(t,r),\quad \forall r\in\mathbb{R},\ t\in(0,T),$$

and assume that

$$\begin{aligned}\gamma_1+\alpha_1|r|^{p_1} &\leq j(t,r)\leq\gamma_2+\alpha_2|r|^{p_2}, && \forall r\in\mathbb{R},\\ j(t,-r) &\leq C_1 j(t,r)+C_2, && \forall r\in\mathbb{R},\end{aligned} \tag{2.9}$$

where $1<p_1\leq p_2<\infty$ and $C_1,\alpha_1,\alpha_2>0$, $\gamma_1,\gamma_2,C_2\in\mathbb{R}$.

We apply Theorem 2.1, where $V=H_0^1(\mathcal{O})\cap L^{p_1}(\mathcal{O})$, $V'=H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O})$, where $H^{-1}(\mathcal{O})=(H_0^1(\mathcal{O}))'$ is the dual of $H_0^1(\mathcal{O})$, and φ is the function

$$\varphi(t,u)=\int_{\mathcal{O}}\left(\frac{1}{2}|\nabla u(\xi)|^2+j(t,u(\xi))\right)d\xi,\quad \forall u\in V,\ t\in(0,T). \tag{2.10}$$

Then, by Theorem 2.1, we obtain that

Corollary 2.1. *Under assumptions* (2.8) *for each* $f \in L^{p_2'}(0,T;H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O}))$ *and* $y_0 \in V$ *there is a unique solution* y *to* (2.8) *which satisfies*

$$y \in L^{p_1}(0,T;H_0^1(\mathcal{O}) \cap L^{p_1}(\mathcal{O})) \cap C([0,T];L^2(\mathcal{O})), \tag{2.11}$$

$$\frac{\partial y}{\partial t} \in L^{p_2'}(0,T;H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O})). \tag{2.12}$$

Similarly, by Theorem 2.2 we have

Corollary 2.2. *Assume that, instead of* (2.8), *the function* j *satisfies the weaker assumption* $j(t,-r) \le C_1 j(t,r) + C_2$ *for all* $r \in \mathbb{R}$ *and*

$$\left\{\begin{array}{rcll} \gamma_1 + \alpha_2 |r|^{p_1} & \le & j(t,r), & \forall r \in \mathbb{R},\ t \in (0,T), \\ j(t,r) & \le & C_M, & \forall |r| \le M,\ \forall M > 0,\ t \in (0,T). \end{array}\right. \tag{2.13}$$

Then, for $f \in L^{p_1'}(0,T;H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O}))$ *and* $y_0 \in V$, *there is a unique solution* y *to* (2.8) *satisfying* (2.11) *and*

$$\frac{\partial y}{\partial t} \in L^1(0,T;H^{-1}(\mathcal{O})+L^{p_1'}(\mathcal{O})). \tag{2.14}$$

The conjugate φ^* to the function φ is given by

$$\varphi^*(t,v) = \sup\left\{(u,v) - \int_{\mathcal{O}} \left(\frac{1}{2}|\nabla u|^2 + j(t,u)\right) d\xi;\ u \in H_0^1(\mathcal{O}))\right\}$$

and, by Fenchel's duality theorem, we have after some calculation (see [5], p. 219)

$$\varphi^*(t,v) = \inf_u \left\{\frac{1}{2}\,\|v+u\|^2_{H^{-1}(\mathcal{O})} + \int_{\mathcal{O}} j^*(t,u)d\xi\right\},$$

which is just the Moreau regularization of the function $u \to \int_\Omega j^*(t,u)d\xi$ in the space $H^{-1}(\mathcal{O})$. Then, by Theorems 2.1 and 2.2 it follows that the solution y given by Corollaries 2.1 and 2.2 are given by

$$y = \arg\min\left\{\int_0^T \left(\varphi(t,u) + \varphi^*\left(t, f - \frac{du}{dt}\right)\right) dt + \frac{1}{2}\int_\Omega u^2(T,\xi)d\xi\right\},$$

where φ, φ^* are as above.

Example 2.2. (The porous media equation) Consider the equation

$$\begin{array}{ll} \dfrac{\partial y}{\partial t} - \Delta\beta(t,y) \ni f & \text{in } (0,T)\times\mathcal{O}, \\ y(0,\xi) = y_0(\xi) & \text{in } \mathcal{O}, \\ \beta(t,y) = 0 & \text{on } (0,T)\times\partial\mathcal{O}, \end{array} \tag{2.15}$$

where $\beta : (0,T)\times\mathbb{R} \to 2^{\mathbb{R}}$ is measurable in t and maximal monotone in $y \in \mathbb{R}$.

Assume that condition (2.8) holds for

$$\begin{aligned} \frac{2d}{d+2} < p_1 &\leq p_2 < \infty \qquad \text{if } d > 2, \\ 1 < p_1 &\leq p_2 < \infty \qquad \text{if } d = 1, 2. \end{aligned} \tag{2.16}$$

We shall apply here Theorem 2.1 for

$$H = H^{-1}(\mathcal{O}), \quad V = L^{p_1}(\mathcal{O})$$

and

$$\varphi(t,y) = \left\{ \begin{array}{ll} \displaystyle\int_{\mathcal{O}} j(t,y(\xi))d\xi \text{ if } y \in H^{-1}(\mathcal{O}) \text{ and } j(t,y) \in L^1(\mathcal{O}), \\ +\infty \qquad \text{otherwise.} \end{array} \right\} \tag{2.17}$$

The space V' is, in this case, the dual of $V = L^{p_1}(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ (by the Sobolev embedding theorem we have this inclusion) with the pivot space $H^{-1}(\mathcal{O})$ endowed with the scalar product $\langle u, v\rangle_{-1} = \int_{\mathcal{O}} u(-\Delta)^{-1}v\,d\xi$, where $D(\Delta) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$. It is easily seen that $\Delta^{-1}V' \subset L^{p_2}(\mathcal{O})$. Then, as easily follows, $\partial\varphi(t,y) = -\Delta\partial j(t,y)$ (see, e.g.,[4], p. 68), we obtain

Corollary 2.3. *Under assumptions* (2.16)*, for each* $y_0 \in L^{p_1}(\mathcal{O})$ *and* $f \in L^{p_1'}(0,T;H^{-1}(\mathcal{O})) \subset L^{p_1'}(0,T;V')$ *there is a unique solution* y *to* (2.15) *such that*

$$y \in L^{p_1}((0,T)\times\mathcal{O}), \ \frac{dy}{dt} \in L^{p_2'}(0,T;V'), \tag{2.18}$$

$$\begin{aligned} \frac{\partial y}{\partial t} = \Delta\eta \ \textit{in } (0,T)\times\mathcal{O}; \quad \eta \in L^{p_2}((0,T)\times\mathcal{O}), \\ \eta \in \beta(y) \ \textit{a.e. in } (0,T)\times\mathcal{O}. \end{aligned} \tag{2.19}$$

Example 2.3. (Parabolic nonlinear BVP of divergence type) Consider the equation

$$\begin{aligned} &\frac{\partial y}{\partial t} - \operatorname{div}\, a(t,\nabla y) = f \qquad \text{in } (0,T)\times\mathcal{O}; \\ &y = 0 \qquad\qquad\qquad\qquad\quad \text{on } (0,T)\times\partial\mathcal{O}. \end{aligned} \tag{2.20}$$

Here $a(t,r) = \partial j(t,r) : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$, where $j(t,\cdot) : \mathbb{R}^d \to \mathbb{R}$ is convex, continuous $y \in \mathbb{R}^d$ and measurable in t. Moreover, there are $\alpha_i > 0$, $\gamma_i \in \mathbb{R}$, $i = 1,2$, and $1 < p_1 \leq p_2 < \infty$, such that

$$\begin{aligned} r_1 + \alpha_1\|r\|_{\mathbb{R}^d}^{p_1} &\leq j(t,r) \leq \gamma_2 + \alpha_2\|r\|_{\mathbb{R}^d}^{p_2}, \quad \forall r \in \mathbb{R}^d, \\ \frac{j(t,r)}{j(t,-r)} &\leq C, \qquad\qquad\qquad\qquad\quad \forall r \in \mathbb{R}. \end{aligned} \tag{2.21}$$

One applies Theorem 2.1 for $V = W_0^{1,p_1}(\mathcal{O})$, $V' = W^{-1,p_1'}(\mathcal{O})$, $H = L^2(\mathcal{O})$ if $p_1 \geq 2$ and $V = W_0^{1,p_1}(\mathcal{O}) \cap L^2(\mathcal{O})$ if $1 < p_1 < 2$.

In this case, $\varphi : (0,T) \times V \to \mathbb{R}$ is defined by

$$\varphi(t,y) = \begin{cases} \displaystyle\int_{\mathcal{O}} j(t, \nabla y(\xi))d\xi & \text{if } j(t,\nabla y) \in L^1(\mathcal{O}), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.22}$$

We obtain

Corollary 2.4. *Under assumptions* (2.21) *for all* $y_0 \in W_0^{1,p_1}(\mathcal{O})$ *and* $f \in L^{p_1'}(0,T;V')$ *there is a unique solution* y *to* (2.20)

$$y \in L^{p_1}(0,T;V) \cap C([0,T];L^2(\mathcal{O})), \tag{2.23}$$

$$\frac{\partial y}{\partial t} \in L^{p_2'}(0,T;V'). \tag{2.24}$$

Remark 2.1. Multivalued functions β arise naturally if one attempts to treat parabolic equations with discontinuous monotone nonlinearities. For instance, the equation

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y + \beta_0(t,y) = f(t,\xi) && \text{in } (0,T) \times \mathcal{O}, \\ &y(0,\xi) = y_0(\xi), && \xi \in \mathcal{O}, \\ &y(t,\xi) = 0, && (t,\xi) \in (0,T) \in \partial\mathcal{O}, \end{aligned}$$

where $r \to \beta_0(t,r)$ is monotonically increasing and discontinuous in $r = r_j$, can be put in the form (2.8), where

$$\beta(t,r) = \begin{cases} \beta_0(t,r), & r \neq r_j, \\ [\beta_0(t,r_j - 0),\ \beta_0(t,r_j+0)], & r = r_j, \end{cases}$$

and for which Corollary 2.1 is applicable.

Multivalued functions $\partial\varphi(t,\cdot)$ arise also in the treatment of parabolic problems with free boundary. For instance, the free boundary problem

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y = f && \text{in } \{y > 0\}, \\ &y \geq 0 && \text{in } (0,T) \times \mathcal{O}, \\ &y(0,\xi) = y_0(\xi) && \text{in } \mathcal{O}, \\ &y = 0 && \text{on } (0,T) \times \partial\mathcal{O}, \end{aligned}$$

can be written in the form (1.1), where $V = H_0^1(\mathcal{O})$, $H = L^2(\mathcal{O})$ and

$$\varphi(t,y) = \begin{cases} \displaystyle\frac{1}{2}\int_{\mathcal{O}} |\nabla y|^2 d\xi & \text{if } y \in K, \\ +\infty & \text{otherwise,} \end{cases}$$

where $K = \{y \in H_0^1(\mathcal{O});\ y \geq 0 \text{ a.e. in } \mathcal{O}\}$. (See, e.g., [4], p. 209.)

Remark 2.2. Equation (2.15) is relevant in dynamics of fluid flows in porous media as well as in that of underground water flows. In this later case, (2.15) reduces to the Richards equation which, in presence of a transport term, is written as

$$\frac{\partial y}{\partial t} - \Delta\beta(t,y) + \operatorname{div}\, K(t,y) = 0.$$

Remark 2.3. In the works [10,13,14] there are several examples of physical problems which are reduced to variational problems by the above procedure, as well as in the recent book [11] by N. Ghoussoub. In particular, in the work [13] the doubly nonlinear equation $\partial\psi\left(\frac{dy}{dt}\right) + \partial\varphi(y) \ni f,\ y(0) = y_0$ is studied via the above Brezis–Ekeland principle.

There are some recent extensions of the Brezis–Ekeland principle to nonlinear equations of the form

$$\frac{dy}{dt} + Ay \ni f, \quad t \in (0,T),\ y(0) = y_0, \tag{2.25}$$

where A is a maximal monotone operator of potential type. This representation of the Cauchy problem (2.25) as a variational problem is via Fitzpatrick function [9]. For a presentation of this approach we refer to the work of A. Visintin [14] (See also the monograph [11].)

3 Proofs

3.1 Proof of Theorem 2.1

Without loss of generality, we may assume that y_0=0. This can be achieved by shifting the initial data y_0 to origin via the transformation $y \to y - y_0$.

For simplicity, we shall write $y' = \frac{dy}{dt}\cdot$

As noticed earlier in Introduction, we have (see (1.6), (1.7))

$$\varphi(t,y(t)) + \varphi^*(t, f(t) - y'(t)) = (f(t) - y'(t), y(t)) \text{ a.e. } t \in (0,T),$$

while

$$\varphi(t,z(t)) + \varphi^*(t, f(t) - z'(t)) - (f(t) - z'(t), z(t)) \geq 0 \text{ a.e. } t \in (0,T),$$

for all $z \in L^{p_1}(0,T;V) \cap W^{1,p_2'}([0,T];V')$. Therefore, we are lead to the optimization problem

$$\begin{aligned}\text{Min}\Big\{ \int_0^T (\varphi(t,y(t)) + \varphi^*(t,f(t)-y'(t)) - (f(t)-y'(t),y(t)))dt;\\ y \in L^{p_1}(0,T;V) \cap W^{1,p_2'}([0,T];V'),\ y(0) = 0\Big\}.\end{aligned} \tag{3.26}$$

However, since the integral $\int_0^T (y'(t),y(t))dt$ might not be well defined, taking into account that (see, e.g., [4], p. 23)

$$\frac{1}{2}\,\frac{d}{dt}\,\|y(t)\|_V^2 = (y'(t),y(t)) \qquad \text{a.e. } t \in (0,T),$$

for each $y \in L^{p_1}(0,T;V) \cap W^{1,p_2'}([0,T];V')$, we shall replace (3.26) by the following convex optimization problem

$$\begin{aligned}\mathrm{Min}\Big\{&\int_0^T(\varphi(t,y(t)))+\varphi^*(t,f(t)-y'(t))-(f(t),y(t))dt+\frac{1}{2}\,\|y(T)\|_V^2;\\ &y \in L^{p_1}(0,T;V)\cap W^{1,p_2'}([0,T];V'),\ y(0)=0, y(T)\in H\Big\},\end{aligned} \tag{3.27}$$

which is well defined because, as easily follows by Hypothesis (ii), we have, by virtue of the conjugacy formulae,

$$\overline{\gamma}_1 + \overline{\alpha}_1\|\theta\|_{V'}^{p_2'} \le \varphi^*(t,\theta) \le \overline{\gamma}_2 + \overline{\alpha}_2\|\theta\|_{V'}^{p_1'},\ \forall \theta \in V' \text{ a.e. } t \in (0,T). \tag{3.28}$$

We prove now that problem (3.27) has a solution y^*, which is also a solution to (1.1). To this end, we set $d^* = \inf\ (3.27)$ and choose a sequence

$$\{y_n\} \subset L^{p_1}(0,T;V) \cap W^{1,p_2'}([0,T];V')$$

such that $y_n(0) = 0$ and

$$\begin{aligned}d^* \le&\int_0^T(\varphi(t,y_n(t)))+\varphi^*(t,f(t)-y_n'(t))-(f(t),y_n(t))dt+\frac{1}{2}\,|y_n(T)|_H^2\\ &\le d^* + \frac{1}{n},\ \forall n \in \mathbb{N}.\end{aligned} \tag{3.29}$$

By Hypothesis (ii) and by (3.28), we see that

$$\|y_n\|_{L^{p_1}(0,T;V)} + \|y_n'\|_{L^{p_2'}(0,T;V')} \le C, \quad \forall n \in \mathbb{N},$$

and, therefore, on a subsequence, we have

$$\begin{array}{rcll} y_n & \to & y & \text{weakly in } L^{p_1}(0,T;V),\\ y_n' & \to & y' & \text{weakly in } L^{p_2'}(0,T;V'),\\ y_n(T) & \to & y(T) & \text{weakly in } H.\end{array} \tag{3.30}$$

Inasmuch as the functions $y \to \int_0^T \varphi(t,y(t))dt,\ z \to \int_0^T \varphi^*(t,f(t)-z'(t))dt$ and $y_1 \to |y_1|_H^2$ are weakly lower-semicontinuous in $L^{p_1}(0,T;V)$, $L^{p_2'}(0,T;V')$ and H, respectively, letting n tend to zero into (3.29), we see that

$$\int_0^T(\varphi(t,y(t)))+\varphi^*(t,f(t)-y'(t))-(f(t),y(t)))dt + \frac{1}{2}\,|y(T)|_H^2 = d^*, \tag{3.31}$$

that is, y is solution to (3.27). Now, we are going to prove that $d^* = 0$. To this aim, we invoke the duality theorem for optimal convex control problems (see [5]). Namely, we have

$$d^* + \min\,(\mathrm{P}_1^*) = 0, \tag{3.32}$$

where (P_1^*) is the dual optimization problem corresponding to (3.27), that is,

$$(\mathrm{P}_1^*)\ \ \mathrm{Min}\Big\{\int_0^T\{\varphi(t,-p(t))+\varphi^*(t,f(t)+p'(t))+(f(t),p(t))\}dt+\frac{1}{2}\,|p(T)|_H^2;\\ p\in L^{p_1'}(0,T;V)\cap W^{1,p_2'}(0,T;V')\Big\}.$$

Clearly, for $p=-y$, we get $\min(\mathrm{P}_1^*)\le d^*$ and so, by (3.32), we see that

$$\min(\mathrm{P}_1^*)\le 0. \tag{3.33}$$

On the other hand, if $\widetilde{p}$ is optimal in (P_1^*), we have

$$(\widetilde{p}',\widetilde{p})\in L^1(0,T),\ \int_0^T(\widetilde{p}',\widetilde{p})dt=\frac{1}{2}\,(|\widetilde{p}(T)|_H^2-\frac{1}{2}\,|\widetilde{p}(0)|_H^2). \tag{3.34}$$

Indeed, we have

$$-(\widetilde{p}'(t),\widetilde{p}(t))\le\varphi(t,-\widetilde{p}(t))+\varphi^*(t,f(t)+p'(t))+(f(t),\widetilde{p}(t))\ \text{a.e. }t\in[0,T]$$

and

$$(\widetilde{p}'(t)+f(t),\widetilde{p}(t))\le\varphi(t,\widetilde{p}(t))+\varphi^*(t,f(t)+\widetilde{p}'(t))\ \text{a.e. }t\in[0,T].$$

Since $\varphi(t,-\widetilde{p})\in L^1(0,T)$, by Hypothesis (iv), it follows that $\varphi(t,\widetilde{p})\in L^1(0,T)$, too, and therefore $(\widetilde{p}',\widetilde{p})\in L^1(0,T)$, as claimed.

Now, since

$$\frac{1}{2}\,\frac{d}{dt}\,|\widetilde{p}(t)|_H^2=(\widetilde{p}'(t),\widetilde{p}(t))\ \text{a.e. }t\in(0,T),$$

we get (3.34), as claimed. This means that

$$\begin{aligned}\min(\mathrm{P}_1^*) &= \int_0^T(\varphi(t,-\widetilde{p}(t))+\varphi^*(t,f(t)+\widetilde{p}'(t))+(f(t)+\widetilde{p}'(t),\widetilde{p}(t))dt\\ &\quad+\frac{1}{2}\,\|\widetilde{p}(0)\|_H^2\ge 0\end{aligned}$$

by virtue of (1.5)–(1.6). Then, by (3.33), we get $d^*=0$, as claimed.

The same relation (3.34) follows for y^* and so,

$$\frac{1}{2}\,(|y^*(t)|_H^2-|y^*(s)|_H^2)=\int_s^t((y^*)'(\tau),y^*(\tau))d\tau,\quad 0\le s\le t\le T.$$

This implies that $y\in C([0,T];H)$ and

$$\frac{1}{2}\,|y^*(T)|^2=\int_0^T((y^*)'(\tau),y^*(\tau))d\tau.$$

Substituting the latter into (3.31), we have that y^* is solution to (3.31) and also that

$$\int_0^T((\varphi(t,y^*(t))+\varphi^*(t,f(t)(y^*)'(t))-(f(t)-(y^*)'(t),y^*(t))=0.$$

Hence,

$$\varphi(t, y^*(t)) + \varphi^*(t, f(t)(y^*)'(t)) - (f(t) - (y^*)'(t), y^*(t)) = 0 \quad \text{a.e } t \in (0, T)$$

and, therefore, $(y^*(t))' + \partial\varphi(t, y^*(t)) \ni f(t)$ a.e. $t \in (0, T)$, as claimed.

The uniqueness of a solution y^* satisfying (1.1) is immediate by monotonicity of $u \to \partial\varphi(t, u)$ because, for two such solutions y_1^*, y_2^*, we have therefore

$$\frac{d}{dt}\, \|y_1^*(t) - y_2^*(t)\|_H^2 \leq 0 \quad \text{a.e. } t \in (0, T)$$

and, since $y_1^* - y_2^*$ is H-valued continuous and $y_1^*(T) - y_2^*(T) = 0$, we infer that $y_1^* - y_2^* \equiv 0$, as claimed. This completes the proof of Theorem 2.1.

3.2 Proof of Theorem 2.2

First we note that, by hypothesis (iii), part (2.2), we have for all $N > 0$

$$\varphi^*(t, v) \geq N\|v\|_{V'} - C_N, \quad \forall v \in V'.$$

This implies that

$$\lim_{\|v\|_{V'} \to \infty} \frac{\varphi^*(t, v)}{\|v\|_{V'}} = +\infty \ \text{ uniformly in } t. \tag{3.35}$$

Now, coming back to (3.29), we see by (2.2) and (3.35) that

$$\|y_n\|_{L^{p_1}(0,T;V)} \leq C, \quad \forall n, \tag{3.36}$$

and, by the Pettis weak compacity theorem in $L^1(0, T; V;)$ (see, e.g., [4]), we have that

$$\{f - y_n'\}_n \text{ is weakly compact in } L^1(0, T; V'. \tag{3.37}$$

Hence, on a subsequence, again denoted n, we have

$$\begin{array}{rcll} y_n & \to & y & \text{weakly in } L^{p_1}(0, T; V), \\ y_n' & \to & y' & \text{weakly in } L^1(0, T; V'). \end{array}$$

Then, letting $n \to \infty$ into (3.29), we see that $y \in W^{1,1}([0, T]; V') \cap L^{p_1}(0, T; V)$ is solution to (3.27), that is, (3.31) holds.

From this point, the proof is identical with that of Theorem 2.1.

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