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On the instability of solutions of seventh order nonlinear delay differential equations

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Abstract. A kind of seventh order nonlinear delay differential equations is considered. By using the Lyapunov-Krasovskii functional approach [5], some sufficient conditions are established which guarantee that the zero solution of the equation considered is unstable. Our conditions are new and supplement previously known results.

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1 Introduction

Since 1992 till now, by using the Lyapunov's direct method, the qualitative behaviors of solutions of the seventh order nonlinear differential equations without a deviating argument have been studied and are still being investigated in the literature. See, for example, the papers of Bereketoğlu [2], Sadek [6], Tejumola [7], Tunç [8, 9], Tunç and Tunç [10]. In the mentioned papers, [6–10], the Lyapunov's direct method was used to show the instability of the solutions of some seventh order nonlinear differential equations without a deviating argument. However, to the best of our knowledge, we did not find any paper relative to the instability of the solutions of the seventh order linear and nonlinear differential equations with a deviating argument in the literature. The basic reason related to the absence of any paper on this topic may be the difficulty of the construction or definition of appropriate Lyapunov functions or functionals for the instability problems relative to the seventh order linear and nonlinear differential equations with a deviating argument.

As regards our problem here, in 2000, Tejumola [7] studied the instability of the zero solution of the seventh order nonlinear differential equation without a deviating argument

$$x^{(7)} + a_1x^{(6)} + a_2x^{(5)} + a_3x^{(4)} + \psi_4(x, x', ..., x^{(6)})x''' + \psi_5(x')x''+ \\ +\psi_6(x, x', ..., x^{(6)}) + \psi_7(x) = 0. \qquad (1)$$

In this paper, instead of Eq. (1), we take into consideration the seventh order nonlinear differential equation with a constant deviating argument r:

$$x^{(7)} + a_1x^{(6)} + a_2x^{(5)} + a_3x^{(4)} + \psi_4(x, x(t-r), x', x'(t-r), ..., x^{(6)}(t-r))x''' +$$

$$+\psi_5(x')x'' + \psi_6(x, x(t-r), x', x'(t-r), ..., x^{(6)}(t-r)) + \psi_7(x(t-r)) = 0. \quad (2)$$

We write Eq. (2) in system form as

$$x_1' = x_2,\ x_2' = x_3,\ x_3' = x_4,\ x_4' = x_5,\ x_5' = x_6,\ x_6' = x_7,$$
$$x_7' = -a_1x_7 - a_2x_6 - a_3x_5 - \psi_4(x_1, x_1(t-r), ..., x_7(t-r))x_4 - \psi_5(x_2)x_3 -$$
$$-\psi_6(x_1, x_1(t-r), ..., x_7(t-r)) - \psi_7(x_1) + \int\limits_{t-r}^{t} \psi_7'(x_1(s))x_2(s)ds, \quad (3)$$

which is obtained as usual by setting $x = x_1,\ x' = x_2,\ x'' = x_3,\ x''' = x_4,\ x^{(4)} = x_5,\ x^{(5)} = x_6$ and $x^{(6)} = x_7$ in (2), where r is a positive constant, $a_1,\ a_2$ and a_3 are some constants, the primes in Eq. (2) denote differentiation with respect to t, $t \in \Re_+$, $\Re_+ = [0, \infty)$; the functions ψ_4, ψ_5, ψ_6 and ψ_7 are continuous on $\Re^{14}$, $\Re$, $\Re^{14}$ and $\Re$ with $\psi_6(x_1, x_1(t-r), 0, ..., x_4(t-r)) = \psi_7(0) = 0$, and satisfy a Lipschitz condition in their respective arguments. Hence, the existence and uniqueness of the solutions of Eq. (2) are guaranteed (see Elśgolt́s [1, p. 14, 15]). We assume in what follows that the function ψ_7 is differentiable, and $x_1(t),\ x_2(t), x_3(t), x_4(t), x_5(t),\ x_6(t)$ and $x_7(t)$ are abbreviated as $x_1,\ x_2, x_3, x_4, x_5,\ x_6$ and x_7, respectively.

Here, by defining an appropriate Lyapunov functional, we prove an instability theorem for Eq. (2). By this work, we improve an instability result obtained in the literature [7, Theorem 6] relative to a seventh order nonlinear differential equation without a deviating argument to the instability of the zero solution of a certain seventh order nonlinear differential equation with a deviating argument, Eq. (2). Our motivation comes from the papers contained in the references of this paper.

Let $r \geqslant 0$ be given, and let $C = C([-r, 0], \Re^n)$ with

$$\|\phi\| = \max_{-r \leqslant s \leqslant 0} |\phi(s)|\,,\ \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r, a] \to \Re^n$ is continuous, $0 < A \leqslant \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t+s), -r \leqslant s \leqslant 0, \quad t \geqslant 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system

$$\dot{x} = F(x_t), \quad x_t = x(t+\theta), \quad -r \leqslant \theta \leqslant 0, \quad t \geqslant 0,$$

where $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from the conditions on Fthat each initial value problem

$$\dot{x} = F(x_t), \quad x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A), 0 < A \leqslant \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition 1. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geqslant 0$. The zero solution is said to be unstable if it is not stable.

Theorem 1. *Suppose there exists a Lyapunov function $V : G \to \Re_+$ such that $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. If either*

(i) *$\dot{V}(\phi) > 0$ for all ϕ in G for which*

$$V[\phi(0)] = \max_{-s \leqslant t \leqslant 0} V[\phi(s)] > 0$$

or

(ii) *$\dot{V}(\phi) > 0$ for all ϕ in G for which*

$$V[\phi(0)] = \min_{-s \leqslant t \leqslant 0} V[\phi(s)] > 0,$$

then the solution $x = 0$ of $\dot{x} = F(x_t)$ is unstable (see Haddock and Ko [3]).

2 Main result

The following theorem is our main result.

Theorem 2. *Together with all the assumptions imposed on the functions $\psi_4, \quad \psi_5, \psi_6$ and ψ_7 in Eq. (2), we assume that there exist constants $a_2 < 0$, $a_7 > 0$, $\delta_0 > 0$ and $\delta > 0$ such that the following conditions hold:*

$$\psi_7(x_1) \neq 0, \quad (x_1 \neq 0), \quad \frac{\psi_7(x_1)}{x_1} \geqslant \delta_0, \ (x_1 \neq 0), \ 0 < \psi_7'(x_1) \leqslant a_7,$$

$$\psi_6(x_1, ..., x_4(t-r)) \neq 0, \ (x_2 \neq 0), \quad \frac{1}{4a_2}\psi_4^2(.) - \frac{\psi_6(.)}{x_2} \geqslant \delta, \ (x_2 \neq 0).$$

Then, the zero solution, $x = 0$, of Eq. (2) is unstable provided that $r < \dfrac{\delta}{a_7}$.

Remark 1. For the proof of the theorem, under the conditions sated in the theorem, it suffices to find that there exists a continuous Lyapunov functional $V = V(x_{1t}, ..., x_{7t})$ which has the following three properties, Krasovskii properties [4], say $(K_1), (K_2)$ and (K_3):

(K_1) In every neighborhood of $(0, 0, 0, 0, 0, 0, 0)$ there exists a point $(\xi_1, ..., \xi_7)$ such that $V(\xi_1, ..., \xi_7) > 0$,

(K_2) the time derivative $\dot{V} = \dfrac{d}{dt}V(x_{1t}, ..., x_{7t})$ along solution paths of (3) is positive semi-definite,

(K_3) the only solution $(x_1, ..., x_7) = (x_1(t), ..., x_7(t))$ of (3) which satisfies $\dfrac{d}{dt}V(x_{1t}, ..., x_{7t}) = 0$ $(t \geqslant 0)$, is the trivial solution $(0, 0, 0, 0, 0, 0, 0)$.

Proof. Consider the Lyapunov functional $V = V(x_{1t}, ..., x_{7t})$ defined by

$$V = x_2x_7 + a_1x_2x_6 + a_2x_2x_5 + a_3x_2x_4 - x_3x_6 - a_1x_3x_5 - a_2x_3x_4 + x_4x_5 -$$

$$-\frac{1}{2}a_3x_3^2 + \frac{1}{2}a_1x_4^2 + \int\limits_0^{x_1} \psi_7(s)ds + \int\limits_0^{x_2} \psi_5(s)sds - \lambda \int\limits_{-r}^{0} \int\limits_{t+s}^{t} x_2^2(\theta)d\theta ds, \qquad (4)$$

where s is a real variable such that the integral $\int\limits_{-r}^{0} \int\limits_{t+s}^{t} x_2^2(\theta)d\theta ds$ is non-negative and λ is a positive constant which will be determined later in the proof.

From (4) it follows that

$$V(0, 0, 0, 0, 0, 0, 0) = 0$$

and

$$V = x_2x_7 + a_1x_2x_6 + a_2x_2x_5 + a_3x_2x_4 - x_3x_6 - a_1x_3x_5 - a_2x_3x_4 + x_4x_5 -$$

$$-\frac{1}{2}a_3x_3^2 + \frac{1}{2}a_1x_4^2 + \int\limits_0^{x_1} \frac{\psi_7(s)}{s}sds + \int\limits_0^{x_2} \psi_5(s)sds - \lambda \int\limits_{-r}^{0} \int\limits_{t+s}^{t} x_2^2(\theta)d\theta ds \geqslant$$

$$\geqslant x_2x_7 + a_1x_2x_6 + a_2x_2x_5 + a_3x_2x_4 - x_3x_6 - a_1x_3x_5 - a_2x_3x_4 + x_4x_5 -$$

$$-\frac{1}{2}a_3x_3^2 + \frac{1}{2}a_1x_4^2 + \frac{1}{2}\delta_0x_1^2 + \int\limits_0^{x_2} \psi_5(s)sds - \lambda \int\limits_{-r}^{0} \int\limits_{t+s}^{t} x_2^2(\theta)d\theta ds.$$

Hence, we get

$$V(\varepsilon, 0, 0, 0, 0, 0, 0) = \frac{1}{2}\delta_0\varepsilon^2 > 0$$

for all sufficiently small ε, $\varepsilon \in \Re$, so that every neighborhood of the origin in the $(x_1, ..., x_7)-$space contains points $(\xi_1, ..., \xi_7)$ such that $V(\xi_1, ..., \xi_7) > 0$.

Let

$$(x_1, ..., x_7) = (x_1(t), ..., x_7(t))$$

be an arbitrary solution of (3).

Differentiating the Lyapunov functional V in (4) along this solution, we get

$$\dot{V} = x_5^2 - a_2x_4^2 - \psi_4(x_1, ..., x_7(t-r))x_2x_4 - \psi_6(x_1, ..., x_7(t-r))x_2 +$$

$$+x_2\int\limits_{t-r}^{t}\psi_7'(x_1(s))x_2(s)ds-\lambda r x_2^2+\lambda\int\limits_{t-r}^{t}x_2^2(s)ds.$$

The assumption $0<\psi_7'(x_1)\leqslant a_7$ of the theorem and the estimate $2|mn|\leqslant m^2+n^2$ imply that

$$x_2\int\limits_{t-r}^{t}\psi_7'(x_1(s))x_2(s)ds\geqslant -|x_2|\int\limits_{t-r}^{t}\psi_7'(x_1(s))\,|x_2(s)|\,ds\geqslant$$

$$\geqslant -\frac{1}{2}a_7 r x_2^2-\frac{1}{2}a_7\int\limits_{t-r}^{t}x_2^2(s)ds.$$

Hence

$$\dot V\geqslant x_5^2-a_2\left[x_4+\frac{1}{2a_2}\psi_4(x_2,...,x_4(t-r))x_2\right]^2+$$

$$+\frac{1}{4a_2}\psi_4^2(x_1,...,x_7(t-r))x_2^2-\psi_6(x_1,...,x_7(t-r))x_2-$$

$$-\{(\lambda+\frac{1}{2}a_7)r\}x_2^2+\left(\lambda-\frac{1}{2}a_7\right)\int\limits_{t-r}^{t}x_2^2(s)ds.$$

Let $\lambda=\frac{1}{2}a_7$. Then, we get

$$\dot V\geqslant x_5^2-a_2\left[x_4+\frac{1}{2a_2}\psi_4(x_1,...,x_7(t-r))x_2\right]^2+$$

$$+\left[\frac{1}{4a_2}\psi_4^2(x_1,...,x_7(t-r))-\frac{\psi_6(x_1,...,x_7(t-r))}{x_2}-a_7 r\right]x_2^2\geqslant$$

$$\geqslant(\delta-a_7 r)x_2^2>0$$

provided that $r<\dfrac{\delta}{a_7}$. Thus if the assumptions of the theorem hold then $\dot V$ is positive semi-definite.

Now observe that $\dot V=0$ for all $t\geqslant 0$ necessarily implies that $x_2=0$ and therefore also that

$$x_2=x'=0,\ x_3=x''=0,\ x_4=x'''=0,$$

$$x_5=x^{(4)}=0,\ x_6=x^{(5)}=0,\ x_7=x^{(6)}=0$$

for all $t\geqslant 0$. Hence

$$x_2=x_3=x_4=x_5=x_6=x_7=0\ (t\geqslant 0).$$

Moreover, in view of $\dot V=0$ and the system (4), one can also easily obtain $x_1=x_2=x_3=x_4=x_5=x_6=x_7=0$, which verifies the property (K_3) of

Krasovskii [4]. It now follows that the Lyapunov functional V thus has all the requisite Krasovskii properties, $(K_1), (K_2)$ and (K_3), subject to the conditions in the theorem. By the above discussion, we conclude that the zero solution of Eq. (2) is unstable. The proof of the theorem is completed. □

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