BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(65), 2011, Pages 28–32
ISSN 1024–7696

Reconstruction of centrally symmetric convex bodies in $\boldsymbol{R}^n$

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Abstract. The article considers the problem of existence and uniqueness of a centrally symmetric convex body in $\mathbb{R}^n$ for which the projection curvature radius function coincides with given function. A necessary and sufficient condition is found that ensures a positive answer. Also we find a representation for the support function of a centrally symmetric convex body.

Mathematics subject classification: 52A20, 53C45, 53C65.
Keywords and phrases: convex geometry, integral geometry, projection curvature.

1 Introduction

Let F be a function defined on 2-dimensional unit sphere S^2. The existence and uniqueness of convex body $\mathbf{B} \subset \mathbb{R}^3$ for which the mean curvature radius at the point on $\partial\mathbf{B}$ with outer normal direction ω coincides with $F(\omega)$ was posed by Christoffel (see [4]). The corresponding problem for Gauss curvature was posed and solved by Minkowski. A. D. Aleksandrov and A. V. Pogorelov generalized these problems for a class of symmetric functions $G(R_1(\omega), R_2(\omega))$ of principal radii of curvatures [4].

Let $\mathbf{B} \subset \mathbb{R}^n$ be a convex body with sufficiently smooth boundary and let $R_1(\omega), \dots, R_{n-1}(\omega)$ signify the principal radii of curvature of the boundary of $\mathbf{B}$ at the point with outer normal direction $\omega \in \mathrm{S}^{n-1}$. In n-dimensional case a Christoffel–Minkowski problem was posed and solved by Firay (see [6]) and Berg (see [8]): what are necessary and sufficient conditions for a function F, defined on S^{n-1} to be the function $\sum R_{i_1}(\omega) \cdots R_{i_p}(\omega)$ for a convex body, where $1 \le p \le n-1$ and the sum is extended over all increasing sequences $i_1, \cdots, i_p$ of indices chosen from the set $i = 1, \dots, n-1$.

In this paper we consider a similar problem posed for the 2-dimensional projection curvature radii of centrally symmetric convex bodies in $\mathbb{R}^n$. We use the following notation. By $\mathcal{B}_o$ we denote the class of convex bodies $\mathbf{B} \subset \mathbb{R}^n$ that have a center of symmetry at the origin $O \in \mathbb{R}^n$. For two different directions $\omega, \xi \in \mathrm{S}^{n-1}$, $\omega \neq \xi$ we denote by $B(\omega, \xi)$ the projection of $\mathbf{B} \in \mathcal{B}_o$ onto the 2-dimensional plane $e(\omega, \xi)$ containing the origin and the directions ω and ξ.

We define $R(\omega, \xi) =$ curvature radius of $\partial B(\omega, \xi)$ at the point whose outer normal direction is ω, and call it 2-dimensional projection curvature radius of the body. Since $R(\omega, \xi_1) = R(\omega, \xi_2)$, where $\omega, \xi_1, \xi_2 \in \mathrm{S}^{n-1}$, are linearly dependent vectors, we assume where necessary that ξ is orthogonal to ω.

Let F be a function defined on the space of "flags" an ordered pairs of orthogonal unit vectors $\{(\omega,\psi) : \omega \in \mathrm{S}^{n-1}, \psi \in \mathrm{S}_\omega\}$. By S_ω we denote the great $(n-2)$-subsphere of S^{n-1} with pole $\omega \in \mathrm{S}^{n-1}$. In integral geometry the concept of a flag was first systematically employed by R. V. Ambartzumian in [1]. In this paper we study:

Problem 1: what are necessary and sufficient conditions for a function F to be the 2-dimensional projection curvature radius function of a centrally symmetric convex body and

Problem 2: reconstruction of that centrally symmetric convex body.

In this paper we find a necessary and sufficient condition on $F(\omega,\psi)$ that ensures a positive answer. Note that the uniqueness (up to parallel shifts) follows from the classical uniqueness result on Christoffel problem.

Also we find a simple representation for the support function of a 2-smooth centrally symmetrical convex body in $\mathbb{R}^n$ in terms of 2-dimensional projection curvature radius function.

Now we describe the main result. Let F be a nonnegative function defined on the ordered pairs of orthogonal unit vectors $\mathfrak{F} = \{(\omega,\psi) : \omega \in \mathrm{S}^{n-1}, \psi \in \mathrm{S}_\omega\}$.

Theorem 1. *A nonnegative n times continuously differentiable function F defined on $\mathfrak{F}$ is the 2-dimensional projection curvature radius function of a centrally symmetric convex body if and only if there is an even continuous function f defined on S^{n-1} such that*

$$F(\omega,\psi) = 2\int_{\mathrm{S}_\omega} |<\psi,u>|^2 f(u)\,\lambda_{n-2}(du), \tag{1}$$

for all $\omega \in \mathrm{S}^{n-1}$ and all $\psi \in \mathrm{S}_\omega$, here λ_{n-2} is the Lebesgue measure on S^{n-2}, $<\cdot,\cdot>$ denotes the Euclidean scalar product.

Note that in [3] the same problem was considered in $\mathbb{R}^3$ and a different necessary and sufficient condition was found.

Radon transform provide a technique for studying the Christoffel problem for centrally symmetric convex bodies. The solution of that problem is of different nature for even and odd values of n (see [8]).

To reconstruct the convex body we find (by means of another method) a simple representation for the support function of a centrally symmetric convex body in terms of 2-dimensional projection curvature radius function.

Theorem 2. *The support function of 2-smooth centrally symmetric convex body $\mathbf{B} \subset \mathbb{R}^n$ has the following representation*

$$H(\xi) = \frac{1}{2\sigma_{n-2}}\int_{\mathrm{S}^{n-1}} \frac{R(\omega,\xi)}{\sin^{n-3}(\widehat{\omega,\xi})}\,\lambda_{n-1}(d\omega), \quad \xi \in \mathrm{S}^{n-1}. \tag{2}$$

Here $\widehat{\omega,\xi}$ is the angle between ω and ξ, $\sigma_{n-2} = \lambda_{n-2}(\mathrm{S}^{n-2})$.

We need the following results from the convexity theory.

2 Preliminaries

It is well known (see [7]) that *the support function of every sufficiently smooth convex body* $\mathbf{B} \in \mathcal{B}_o$ *has the unique representation*

$$H(\xi) = \int_{\mathrm{S}^{n-1}} |< \xi, \Omega >| \, h(\Omega) \, \lambda_{n-1}(d\Omega), \quad \xi \in \mathrm{S}^{n-1} \tag{3}$$

with some unique even continuous function $h(\Omega)$ *defined on* S^{n-1}*, not necessarily nonnegative, called the generating density of the body.*

R. Schneider (see [8]) has showed that the smoothness of order n yields the representation with a continuous generating density.

Below we use the following result of N. F. Lindquist (see [8]).
An even continuous function h *defined on* S^{n-1} *is the generating density of a convex body* $\mathbf{B} \in \mathcal{B}_o$ *if and only if*

$$\int_{\mathrm{S}_\omega} |< \psi, u >|^2 \; h(u) \, \lambda_{n-2}(du) \geq 0, \tag{4}$$

for all $\omega \in \mathrm{S}^{n-1}$ *and all* $\psi \in \mathrm{S}_\omega$*.*

The author of the present paper gave a clear geometrical interpretation for integral (4). In [2] has proved the following theorem (here we present a short version of the proof for completeness).

Theorem 3. *For any sufficiently smooth convex body* $\mathbf{B} \in \mathcal{B}_o$

$$R(\omega, \xi) = 2 \int_{\mathrm{S}_\omega} |< \xi, u >|^2 \; h(u) \, \lambda_{n-2}(du), \tag{5}$$

where ξ, $\omega \in \mathrm{S}^{n-1}$, $\xi \perp \omega$, $h(u)$ *is the generating density of* $\mathbf{B}$.

Proof. We need some special representation for the element of Lebesgue measure on S^{n-1}. Given an orthonormal system of unit vectors $z_1, z_2, x_1, x_2, ..., x_{n-2}$ in $\mathbb{R}^n$, we represent $\omega \in \mathrm{S}^{n-1}$ as $\omega = (\nu, \varphi, u)$, where ν is the angle between ω and $e(z_1, z_2)$, φ is the angle between z_1 and the projection of ω onto $e(z_1, z_2)$, while u is the direction of the projection of ω onto the $(n-2)$-dimensional subspace containing $x_1, x_2, ..., x_{n-2}$. The corresponding Jacobian for $n \geq 4$ is (see [6])

$$\lambda_{n-1}(d\omega) = \sin^{n-3} \nu \, \cos \nu \, d\nu \, d\varphi \, \lambda_{n-3}(du). \tag{6}$$

The support function of $B(\omega, \xi)$ is the restriction of $H(\xi)$ (the support function of the body) onto the circle $\mathrm{S}^{n-1} \cap e(\omega, \xi)$. We consider some orthonormal system of unit vectors $z_1, z_2, x_1, ..., x_{n-2}$, where $z_1 = \omega$, $z_2 = \xi$. Let ϕ be the angle between direction $\omega(\phi)$ in $e(\omega, \xi)$ and ω. We have $\omega(\phi) = (\cos \phi, \sin \phi, 0, ..., 0)$. According to the formula for curvature radius in 2-dimensional case (see [5]) we have

$$R(\omega, \xi) \; = H(0) + H''(\phi) \mid_{\phi = 0}, \tag{7}$$

where $H(\phi) = H(\omega(\phi))$. Using (3) we get

$$H(\phi) = \int_{S^{n-1}} |< \omega(\phi), \Omega >| \; h(\Omega)\, d\Omega \; = 2 \int_{\{\Omega\,(\omega,\Omega)\geq 0\}} (\Omega_1 \cos\phi + \Omega_2 \sin\phi)\, h(\Omega)\, d\Omega, \tag{8}$$

where $\Omega = (\Omega_1, \Omega_2, ..., \Omega_n)$. Now we represent Ω as $\Omega = (\nu, \varphi, \delta)$, where $\delta \in S^{n-3}$, ν is the angle between Ω and $e(\omega, \xi)$, and φ is a direction in $e(\varphi, \xi)$. Using (6) for the second derivative we get

$$H''(\phi) = 2 \int_{\{\Omega\,(\omega,\Omega)\geq 0\}} (-\Omega_1 \cos\phi - \Omega_2 \sin\phi)\, h(\Omega)\, d\Omega + \tag{9}$$

$$+2 \int_{S_{\omega(\phi)}} (-\Omega_1 \sin\phi + \Omega_2 \cos\phi)\, h(\nu, \phi + \frac{\pi}{2}, \delta)\; \sin^{n-3}\nu\; \cos\nu\, d\nu \lambda_{n-3}(d\delta).$$

Substituting (9) into (7) and taking into account that $\sin^{n-3}\nu\, d\nu\, \lambda_{n-3}(d\delta) = \lambda_{n-2}(du)$ where $u = (\nu, \delta)$, $u \in S_\omega$ and $\Omega_2 = \cos\nu = \cos(u, \xi)$ we get (5). □

3 Proofs of Theorems 1 and 2

Proof of Theorem 1. *Necessity:* let $R(\omega, \psi)$ be the projection curvature radius of a convex body $\mathbf{B} \in \mathcal{B}_o$. We have to prove that there is an even function f defined on S^{n-1} such that condition (1) satisfies for $R(\omega, \psi)$. It follows from (3) that for a sufficiently smooth convex body the generating density exists. As a function f, we take the generating density of centrally symmetric convex body $\mathbf{B}$. It follows from Theorem 3 that equation (1) is satisfied.

Sufficiency: let F be a nonnegative function defined on $\mathfrak{F}$ for which there is an even continuous function f defined on S^{n-1} such that

$$F(\omega, \psi) = 2 \int_{S_\omega} | < \psi, u > |^2\, f(u)\, \lambda_{n-2}(du), \tag{10}$$

for all $\omega \in S^{n-1}$ and all $\psi \in S_\omega$. Since F is nonnegative the right hand side of (10) is nonnegative. Hence according to Theorem 3 there exists a centrally symmetric convex body $\mathbf{B}$ for which even function f is the generating density of $\mathbf{B}$. It follows from Theorem 3 that the right hand side of (10) is the 2-dimensional projection curvature radius function of $\mathbf{B}$. Hence F is the 2-dimensional projection curvature radius of $\mathbf{B}$. □

Proof of Theorem 2. Let $u \in S_\xi$ be a direction perpendicular to $\xi \in S^{n-1}$. We approximate $B(u, \xi) \subset e(\omega, \xi)$ from inside by polygons that have their vertices on $\partial B(u, \xi)$. We denote by a_i sides of the approximation polygon, by ν_i (ν_i is the angle between the normal direction and ξ) the direction normal to a_i within $e(u, \xi)$. Let $H_{B(u,\xi)}$ be the support function of $B(u, \xi)$. We have

$$4H(\xi) = 4H_{B(u,\xi)}(\xi) = \lim \sum_i |a_i|\; \sin(\widehat{\xi, \nu_i}) = \tag{11}$$

$$= \lim \sum_i R_u(\nu_i)|\nu_{i+1} - \nu_i| \, \sin(\widehat{\xi, \nu_i}) = 2 \int_0^\pi R_u(\nu) \, \sin \nu \, d\nu,$$

$R_u(\nu)$ is the radius of curvature of $\partial B(u, \xi)$ at the point with normal ν. Integrating both sides of (11) in $\lambda_{n-2}(du)$ over S_ξ, and using standard formula $\lambda_{n-1}(d\omega) = \sin^{n-2}\nu \, d\nu \, \lambda_{n-2}(du)$, where $\omega = (\nu, u)$ we obtain

$$2\sigma_{n-2} H(\xi) = \int_{\mathrm{S}^\xi} \int_0^\pi R_u(\nu) \sin \nu \, d\nu \, \lambda_{n-2}(du) =$$

$$= \int_{\mathrm{S}_\xi} \int_0^\pi \frac{R_u(\nu)}{\sin^{n-3} \nu} \sin^{n-2} \nu \, d\nu \, \lambda_{n-2}(du) = \int_{\mathrm{S}^{n-1}} \frac{R(\omega, \xi)}{\sin^{n-3}(\widehat{\omega, \xi})} \lambda_{n-1}(d\omega). \qquad \square$$

Note that replacing $2H(\cdot)$ by the width function $W(\cdot)$ in (2) we get a formula for the width function for all convex bodies (not only centrally symmetric).

I would like to express my gratitude to Professor R. V. Ambartzumian for helpful discussions.

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Received October 27, 2009