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Topological rings with at most two nontrivial closed ideals

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Abstract. In this paper, we describe the Hausdorff topological rings with identity in which every nontrivial closed ideal is topologically maximal, respectively, strongly topologically maximal, and the Hausdorff topological rings with identity which have no more than two nontrivial closed ideals.

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Introduction

In [4], F. Perticani determined the structure of (discrete) commutative rings with identity in which every nontrivial ideal (i. e., distinct from the zero ideal and the whole ring) is maximal. He proved that such a ring, E, has at most two distinct nontrivial ideals, and if E is not simple, then either it is isomorphic to a product of two fields or it is obtained as extension of a one-dimensional vector space over some field, considered as ring with zero multiplication, by the same field in such a way that the mentioned vector space structure coincides with the structure determined by the exact sequence defining the corresponding extension.

We consider here analogous questions in the more general context of topological rings. To be precise, we describe the (not necessarily commutative) topological rings with identity in which every nontrivial closed ideal is topologically maximal, respectively, strongly topologically maximal. We also determine the topological rings with identity which have no more than two nontrivial closed ideals.

Throughout the paper, all topological rings considered are assumed to be Hausdorff. If E is a topological ring and A is an ideal of E, we denote by $\overline{A}$ the closure of A in E, by $ann_E(A)$ the annihilator of A in E, and by $ann^l_E(A)$ and $ann^r_E(A)$ the left annihilator and the right annihilator of A in E, respectively. If B is a closed ideal of E satisfying $A \subset B$, we denote by $ann_E(B/A)$ the annihilator of the quotient E-bimodule B/A in E. Also, the symbol $\cong$ stands for topological isomorphism.

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1 Topological rings in which every nontrivial closed ideal is topologically maximal

As mentioned in Introduction, F. Perticani described in his paper [4] the commutative rings with identity in which every nontrivial ideal is maximal. The purpose of the present paper is to extend the results obtained in [4] to topological rings. We begin by introducing, for topological rings, the analogue of the notion of maximal ideal.

Definition 1. Let E be a topological ring. A closed ideal M of E is said to be topologically maximal if M is proper (i. e., $M \neq E$) and for every closed ideal C of E such that $M \subset C$, either $C = M$ or $C = E$.

Definition 2. A topological ring E is said to be topologically simple in case E is nonzero and has no nontrivial closed (two-sided) ideals.

We will need the following analogue of the well known characterization of maximal ideals.

Lemma 1. *Let E be a topological ring. A closed ideal M of E is topologically maximal if and only if E/M is topologically simple.*

Proof. Let M be a closed ideal of E, and let π denote the canonical projection of E onto E/M.

If M is topologically maximal and if C' is a closed ideal of E/M, then $\pi^{-1}(C')$ is a closed ideal of E and $M \subset \pi^{-1}(C')$, so that $\pi^{-1}(C')$ coincides with either M or E. As $C' = \pi(\pi^{-1}(C'))$, it follows that C' coincides with either the zero ideal or the whole ring E/M.

For the converse, let C be a closed ideal of E such that $M \subset C$. Then $(E/M) \setminus \pi(C) = \pi(E \setminus C)$. Since π is open, it follows that $\pi(C)$ is closed in E/M, and hence $\pi(C)$ coincides with either the zero ideal or E/M. As $C = \pi^{-1}(\pi(C))$, we conclude that either $C = M$ or $C = E$. $\square$

We proceed now to study the structure of topological rings in which every nontrivial closed ideal is topologically maximal.

Lemma 2. *Let E be a topological ring in which every nontrivial closed ideal is topologically maximal. If A and B are different nontrivial closed ideals of E, then $\overline{A+B} = E$ and $A \cap B = \{0\}$.*

Proof. Since A and B are contained in $\overline{A+B}$, the relation $\overline{A+B} \neq E$ would imply $A = \overline{A+B} = B$, because A and B have to be topologically maximal. Similarly, since $A \cap B$ is contained in A and in B, the relation $A \cap B \neq \{0\}$ would imply $A = A \cap B = B$, because $A \cap B$ has to be topologically maximal. $\square$

Lemma 3. *Let E be a topological ring with identity, and let A and B be nontrivial closed ideals of E such that $\overline{A+B} = E$ and $A \cap B = \{0\}$. Then $ann_E(A) = B$ and $ann_E(B) = A$.*

Proof. Since AB and BA are contained in $A \cap B$, we have $A \subset ann_E(B)$ and $B \subset ann_E(A)$. To show the inverse inclusions, pick any $u \in ann_E(B)$ and $v \in ann_E(A)$. Since $\overline{A+B} = E$, we can write $1 = \lim_{\lambda \in L}(a_\lambda + b_\lambda)$, where $(a_\lambda)_{\lambda \in L}$ is a net in A and $(b_\lambda)_{\lambda \in L}$ is a net in B [2, Proposition 1.6.3.]. It follows that

$$u = u \lim_{\lambda \in L}(a_\lambda + b_\lambda) = \lim_{\lambda \in L} ua_\lambda \in A$$

and

$$v = v \lim_{\lambda \in L}(a_\lambda + b_\lambda) = \lim_{\lambda \in L} vb_\lambda \in B.$$

Consequently, $ann_E(A) = B$ and $ann_E(B) = A$. □

With these preparations, we have

Theorem 1. *A topological ring with identity in which every nontrivial closed ideal is topologically maximal cannot have more than two different nontrivial closed ideals.*

Proof. Let E be a topological ring with identity in which every nontrivial closed ideal is topologically maximal, and assume A, B and C are different nontrivial closed ideals of E. By Lemma 2, we have $\overline{A+B} = E$, so that $1 = \lim_{\lambda \in L}(a_\lambda + b_\lambda)$, where $(a_\lambda)_{\lambda \in L}$ is a net in A and $(b_\lambda)_{\lambda \in L}$ is a net in B. Pick any nonzero $c \in C$. The multiplication by c being continuous, it follows that

$$c = c \cdot \lim_{\lambda \in L}(a_\lambda + b_\lambda) = \lim_{\lambda \in L} c \cdot (a_\lambda + b_\lambda) \in \overline{C \cdot A + C \cdot B}.$$

But $C \cdot A \subset C \cap A$ and $C \cdot B \subset C \cap B$. Since $C \cap A = \{0\} = C \cap B$ by Lemma 2 and since E is Hausdorff, this proves that $\overline{CA + CB} = \{0\}$, so $c = 0$, a contradiction. Consequently, E cannot have more than two different nontrivial closed ideals. □

Next we consider the case of topological rings with two nontrivial closed ideals.

Theorem 2. *Let E be a topological ring with identity having two different nontrivial closed ideals. The following statements are equivalent:*

(i) *E has exactly two different nontrivial closed ideals, and these ideals are not comparable with respect to inclusion.*

(ii) *Every nontrivial closed ideal of E is topologically maximal.*

(iii) *There exist two different nontrivial closed ideals A, B of E such that the following conditions hold:*

  (1) *$\overline{A+B} = E$ and $A \cap B = \{0\}$;*

  (2) *A and B are topologically simple rings.*

Proof. Clearly, (i) implies (ii). Assume (ii), and let A and B be different nontrivial closed ideals of E. The fact that condition (1) of (iii) is satisfied follows from Lemma 2. In particular, we can write $1 = \lim_{\lambda \in L}(a_\lambda + b_\lambda)$, where $(a_\lambda)_{\lambda \in L}$ is a net in A and $(b_\lambda)_{\lambda \in L}$ is a net in B. It also follows from Lemma 3 that $ann_E(A) = B$ and $ann_E(B) = A$. To see that A is a topologically simple ring, let I be an arbitrary nonzero closed ideal of A. For any $x \in E$ and $y \in I$, we have

$$xy = x[\lim_{\lambda \in L}(a_\lambda + b_\lambda)]y = x(\lim_{\lambda \in L} a_\lambda y) = \lim_{\lambda \in L}(xa_\lambda)y \in I$$

and

$$yx = y[\lim_{\lambda \in L}(a_\lambda + b_\lambda)]x = (\lim_{\lambda \in L} ya_\lambda)x = \lim_{\lambda \in L} y(a_\lambda x) \in I,$$

so I is an ideal of E. In view of (ii), we must have $I = A$. The proof that B is a topologically simple ring is similar, so condition (2) of (iii) also holds.

Assume (iii). The ideals A and B, whose existence is claimed in (iii), cannot be comparable with respect to inclusion because $A \cap B = \{0\}$. It also follows from Lemma 3 that $ann_E(A) = B$ and $ann_E(B) = A$. To see that A and B are the unique different nontrivial closed ideals of E, pick an arbitrary closed ideal C of E. Then $A \cap C$ is a closed ideal of A and $B \cap C$ is a closed ideal of B. Since A and B are topologically simple rings, it follows that $A \cap C$ coincides with either $\{0\}$ or A and $B \cap C$ coincides with either $\{0\}$ or B. We distinguish cases. If $A \cap C = A$ and $B \cap C = B$, we have $A \subset C$ and $B \subset C$, so that $E = \overline{A + B} \subset C$, and hence in this case $C = E$. Next assume $A \cap C = \{0\}$ and $B \cap C = \{0\}$. Since $AC,\ CA \subset A \cap C$, we have $AC = \{0\} = CA$, so that $C \subset ann_E(A) = B$. In a similar way, $C \subset ann_E(B) = A$. As $A \cap B = \{0\}$, it follows that in this case $C = \{0\}$. Now assume $A \cap C = \{0\}$ and $B \cap C = B$. As we have seen, the relation $A \cap C = \{0\}$ gives $C \subset B$. Since the relation $B \cap C = B$ gives $B \subset C$, it follows that in this case $C = B$. Finally, if $A \cap C = A$ and $B \cap C = \{0\}$, we get in a similar way $C = A$. Consequently, E admits only two different nontrivial closed ideals, namely A and B. □

In view of Theorem 2, it would be interesting to know when a topological ring with exactly two nontrivial closed ideals is topologically isomorphic to the direct product of those ideals. To answer this question, we need a new

Definition 3. Let E be a topological ring and M a closed ideal of E. We say M is strongly topologically maximal if M is topologically maximal and if for any closed ideal C of E, $M + C$ is closed in E.

Lemma 4. *Let E be a topological ring. A proper closed ideal M of E is strongly topologically maximal if and only if for each closed ideal C of E such that $C \not\subset M$ one has $M + C = E$.*

Proof. Assume M is strongly topologically maximal, and let C be an arbitrary closed ideal of E such that $C \not\subset M$. Since $M + C$ is closed in E and properly contains M, we must have $M + C = E$.

Assume the converse. Given an arbitrary closed ideal C of E, we then have $M + C = M$ if $C \subset M$ and $M + C = E$ if $C \not\subset M$, so that $M + C$ is closed in E. It is also clear that M is topologically maximal. □

We have the following

Theorem 3. *Let E be a topological ring with identity having two different nontrivial closed ideals A and B. Every nontrivial closed ideal of E is strongly topologically maximal if and only if A and B are topologically simple rings, and $E \cong A \times B$.*

Proof. If every nontrivial closed ideal of E is strongly topologically maximal, it follows from Theorem 2 that $A \cap B = \{0\}$, $\overline{A + B} = E$, and A, B are topologically simple rings. Further, $\overline{A + B} = A + B$ by Lemma 4, and hence $E \cong A \times B$ by [1, Ch. III, §6, Exer. 6].

Now assume that A and B are topologically simple rings, and that there is an isomorphism of topological rings $h : E \to A \times B$. Set $A' = h^{-1}(A \times \{0\})$ and $B' = h^{-1}(\{0\} \times B)$. It follows that $A' + B' = E$ and $A' \cap B' = \{0\}$, so that, by Theorem 2, A' and B' are the only nontrivial closed ideals of E. In particular $\{A', B'\} = \{A, B\}$. If C is an arbitrary closed ideal of E such that $C \not\subset A$, then C coincides with either B or E, so that $A + C = E$, and hence A is strongly topologically maximal by Lemma 4. Clearly, the same holds also for B. □

2 Topological bimodule structures induced by ideal extensions

Let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an exact sequence of abstract rings and homomorphisms of rings, that is such that $\ker(\varphi) = \{0\}$, $\mathrm{im}(\varphi) = \ker(\psi)$, and $\mathrm{im}(\psi) = B$. As is well known (see [3] or [4]), if $A^2 = \{0\}$, then A can be given a bimodule structure over B.

We establish here a topological version of this fact.

Definition 4. Let A and B be arbitrary topological rings. A topological ring E is said to be an ideal extension of A by B if there exist continuous ring homomorphisms $\varphi : A \to E$ and $\psi : E \to B$ such that the following conditions hold:

(i) φ is injective and open onto its image;

(ii) ψ is surjective and open;

(iii) $\mathrm{im}(\varphi) = \ker(\psi)$.

If, in addition, E has an identity, then it is called a unital ideal extension of A by B.

Clearly, if $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ is a unital ideal extension of A by B, then B has an identity too and ψ is unital.

As usual, when we want to emphasize explicitly the homomorphisms $\varphi : A \to E$ and $\psi : E \to B$ making E an ideal extension of A by B, we identify E with the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

Lemma 5. *Let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an ideal extension of A by B. Then $ann_A(A)$ can be turned into a topological bimodule over B.*

Proof. The multiplication of E determines a B-bimodule structure on $ann_A(A)$ in the following way. Let $a \in ann_A(A)$ and $b \in B$ be arbitrary. Since ψ is surjective, there is $c \in E$ such that $b = \psi(c)$. Then $\varphi(a)c$ and $c\varphi(a)$ belong to $\varphi(A)$ because $\varphi(A)$ is an ideal of E. Given any $x \in A$, we have $(\varphi(a)c)\varphi(x) = \varphi(a)(c\varphi(x)) = 0$ and $\varphi(x)(\varphi(a)c) = \varphi(xa)c = \varphi(0)c = 0$, so that in fact $\varphi(a)c \in ann_{\varphi(A)}\big(\varphi(A)\big)$. Similarly, $c\varphi(a) \in ann_{\varphi(A)}\big(\varphi(A)\big)$. Set $ab = \varphi^{-1}\big(\varphi(a)c\big)$ and $ba = \varphi^{-1}\big(c\varphi(a)\big)$. To see that the products ab and ba are well defined, let c' be another element in E such that $\psi(c') = b$. Then $c - c' \in \ker(\psi) = \operatorname{im}(\varphi)$, and since $a \in ann_A(A)$ and hence $\varphi(a) \in ann_{\varphi(A)}\big(\varphi(A)\big)$, we have $\varphi(a)(c - c') = 0 = (c - c')\varphi(a)$. Consequently, ab and ba are well defined. It is now easy to see that $ann_A(A)$ is a bimodule over B, with respect to its addition induced from A and scalar multiplications defined above. Moreover, the addition is, clearly, continuous.

Let us show that the left scalar multiplication is continuous. The case of the right scalar multiplication is similar. Fix any elements $a \in ann_A(A)$ and $b \in B$, and any neighbourhood V of zero in A. Also choose $c \in E$ such that $\psi(c) = b$. Since φ is open onto its image, $\varphi(V)$ is a neighbourhood of zero in $\varphi(A)$. Now, since $\varphi(A)$ is a topological left E-module, there exist a neighbourhood U of zero in E and a neighbourhood W of zero in $\varphi(A)$ such that

$$UW \subset \varphi(V), \quad U\varphi(a) \subset \varphi(V) \quad \text{and} \quad cW \subset \varphi(V).$$

As φ is continuous and ψ is open, $\varphi^{-1}(W)$ is a neighbourhood of zero in A and $\psi(U)$ is a neighbourhood of zero in B. By the definition of the left scalar multiplication, we then have

$$\psi(U)\big(\varphi^{-1}(W) \cap ann_A(A)\big) \subset V \cap ann_A(A), \qquad \psi(U)a \subset V \cap ann_A(A)$$

and

$$b\big(\varphi^{-1}(W) \cap ann_A(A)\big) \subset V \cap ann_A(A),$$

so the left scalar multiplication $(\beta, \alpha) \to \beta\alpha$ from $B \times ann_A(A)$ to $ann_A(A)$ is continuous at $(0, 0)$, and the mappings $\beta \to \beta a$ from B to $ann_A(A)$ and $\alpha \to b\alpha$ from $ann_A(A)$ to $ann_A(A)$ are continuous at 0. Since a and b were arbitrary, it follows from [5, (2.16)] that the left scalar multiplication is continuous. □

Definition 5. The topological B-bimodule structure of $ann_A(A)$ described above will be referred to as the topological B-bimodule structure determined by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

Corollary 1. *Let A and B be topological rings, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an ideal extension of A by B.*

(i) *If A contains a closed ideal K such that $(A/K)^2 = \{0\}$, then A/K can be turned into a topological bimodule over B.*

(ii) *If B_0 is a closed ideal of B such that $\psi^{-1}(B_0)^2 = \{0\}$, then $\psi^{-1}(B_0)$ can be turned into a topological bimodule over B/B_0.*

Proof. (i) Let $\lambda : A \to A/K$ and $\varrho : E \to E/K$ be the canonical projections. As is well known, there exist continuous ring homomorphisms $\hat{\varphi} : A/K \to E/K$ and $\hat{\psi} : E/K \to B$ such that $\varrho \circ \varphi = \hat{\varphi} \circ \lambda$ and $\psi = \hat{\psi} \circ \varrho$. Moreover, $\hat{\varphi}$ and $\hat{\psi}$ are open onto their images,

$$\ker(\hat{\varphi}) = \ker(\varrho \circ \varphi)/K = \{0\}, \qquad \operatorname{im}(\hat{\psi}) = \operatorname{im}(\psi)$$

and

$$\ker(\hat{\psi}) = \ker(\psi)/K = \varphi(A)/K = \operatorname{im}(\hat{\varphi}).$$

Consequently, the homomorphisms $\hat{\varphi} : A/K \to E/K$ and $\hat{\psi} : E/K \to B$ make E/K an ideal extension of A/K by B. Since $(A/K)^2 = \{0\}$, it follows from Lemma 5 that A/K can be given a topological bimodule structure over B.

(ii) Let $\eta : \psi^{-1}(B_0) \to E$ be the canonical injection of $\psi^{-1}(B_0)$ into E and $\pi : B \to B/B_0$ the canonical projection of B onto B/B_0. Then η and $\pi \circ \psi$ are open onto their images, η is injective, $\pi \circ \psi$ is surjective, and $\operatorname{im}(\eta) = \psi^{-1}(B_0) = \ker(\pi \circ \psi)$, so that η and $\pi \circ \psi$ transform E into an ideal extension of $\psi^{-1}(B_0)$ by B/B_0. By Lemma 5, $\psi^{-1}(B_0)$ can be given a topological bimodule structure over B/B_0. □

Definition 6. The topological B-bimodule structure of A/K described above is referred to as the topological B-bimodule structure determined on A/K by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

Similarly, the topological B/B_0-bimodule structure of $\psi^{-1}(B_0)$ described above is referred to as the topological B/B_0-bimodule structure determined on $\psi^{-1}(B_0)$ by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

Lemma 6. *Let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an ideal extension of A by B. Then $ann^l_A(A)$ can be turned into a topological right B-module. Similarly, $ann^r_A(A)$ can be turned into a topological left B-module.*

Proof. The multiplication by scalars in $ann^l_A(A)$ (respectively, $ann^r_A(A)$) is given by $ab = \varphi^{-1}(\varphi(a)c)$ (respectively, $ba = \varphi^{-1}(c\varphi(a))$) for $a \in ann^l_A(A)$ (respectively, $a \in ann^r_A(A)$), $b \in B$, and $c \in E$ with $b = \psi(c)$. □

Definition 7. The topological B-module structure of $ann^l_A(A)$ (respectively, $ann^r_A(A)$) described above is referred to as the topological B-module structure determined on $ann^l_A(A)$ (respectively, $ann^r_A(A)$) by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$.

3 Topological rings with only one nontrivial closed ideal

In this section, we relate the study of topological rings with only one nontrivial closed ideal to an extension problem, although the cohomology theory for topological rings is not constructed yet.

We use the following simple

Lemma 7. *Let E be a ring and A a nonzero ideal of E. If*

$$ann^l_E(A) = \{0\} = ann^r_E(A),$$

then for any nonzero $x \in A$, AxA is nonzero.

Proof. Pick any nonzero $x \in A$. Since $x \notin ann^r_E(A)$, there exists $a \in A$ such that $ax \neq 0$. Similarly, since $ax \notin ann^l_E(A)$, there exists $a' \in A$ such that $axa' \neq 0$. Hence $AxA \neq \{0\}$. $\square$

Definition 8. Let E be a topological ring. A topological module (respectively, bimodule) A over E is said to be topologically simple in case A is nonzero and has no nontrivial closed submodules (respectively, subbimodules).

Theorem 4. *Let E be a topological ring with identity having only one nontrivial closed ideal A. Then E/A is a unital topologically simple ring, and E can be viewed as an ideal extension $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$ of A by E/A, where η is the canonical injection and π is the canonical projection, such that exactly one of the following conditions hold:*

(i) *$ann_E(A) = \{0\}$ and A is a topologically simple ring;*

(ii) *$ann_E(A) = A$ and A, with the structure given by the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, is a unital topologically simple E/A-bimodule.*

Proof. Consider the natural exact sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, where η is the canonical injection and π is the canonical projection. As is well known, η and π are continuous and open onto their images. Now, since A is the only nontrivial closed ideal of E, it is clear that E/A is a unital topologically simple ring. Further, since E has an identity, we cannot have $ann_E(A) = E$, so that either $ann_E(A) = \{0\}$ or $ann_E(A) = A$. Assume the former, and consider the one-sided annihilators $ann^l_E(A)$ and $ann^r_E(A)$. Clearly, $ann^l_E(A)$ and $ann^r_E(A)$ are closed ideals of E. As in the case of $ann_E(A)$, we have $ann^l_E(A) \neq E$ and $ann^r_E(A) \neq E$. On the other hand, either of equalities $ann^l_E(A) = A$ or $ann^r_E(A) = A$ implies $ann_E(A) = A$, in contradiction with our assumption that $ann_E(A) = \{0\}$. Therefore we must have $ann^l_E(A) = \{0\} = ann^r_E(A)$. To see that A is a topologically simple ring, pick an arbitrary nonzero closed ideal B of A and any nonzero element $b \in B$. By Lemma 7, $\overline{AbA}$ is a nonzero closed ideal of A satisfying $\overline{AbA} \subset B$. Since A is an ideal of E, it then follows that $\overline{AbA}$ is a nonzero closed ideal of E, whence $\overline{AbA} = A$, so $B = A$. Consequently, A is a topologically simple ring, and hence in this case we are led to (i).

Now consider the latter case when $ann_E(A) = A$. By using the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, it follows from Lemma 5 that A can be turned into a topological bimodule over E/A. Moreover, if $a \in A$, then $a \cdot \pi(1) = a \cdot 1 = a$ and $\pi(1) \cdot a = 1 \cdot a = a$, so this bimodule is unital. Pick an arbitrary nonzero closed E/A-subbimodule C of A. Taking into account the definition of scalar multiplications, we see that for any $x \in E$,

$$xC = \pi(x)C \subset C \qquad \text{and} \qquad Cx = C\pi(x) \subset C,$$

so C is a nonzero closed ideal of E contained in A, whence $C = A$. Since the E/A-subbimodule C was picked arbitrarily, it follows that A is a topologically simple E/A-bimodule, and hence in this case we have (ii). $\square$

We next show that the converse is also true.

Theorem 5. *Let A and B be topologically simple rings, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be an ideal extension of A by B. If $ann_E(\varphi(A)) = \{0\}$, then E has only one nontrivial closed ideal, namely $\varphi(A)$.*

Proof. Clearly, $\varphi(A)$ is a nontrivial closed ideal of E. Moreover, if K is a closed ideal of E such that $\varphi(A) \subset K$, then $B \setminus \psi(K) = \psi(E \setminus K)$ is open in B, so that $\psi(K)$ is closed in B. Since B is topologically simple, it follows that either $\psi(K) = \{0\}$ or $\psi(K) = B$, and hence either $K = A$ or $K = E$. Consequently, the ideal $\varphi(A)$ is topologically maximal.

Now, let C be an arbitrary closed ideal of E. Then $\varphi(A) \cap C$ is a closed ideal of $\varphi(A)$. If $\varphi(A) \cap C \neq \{0\}$, we must have $\varphi(A) \cap C = \varphi(A)$ because $\varphi(A)$ is a topologically simple ring. It follows that $\varphi(A) \subset C$, and hence C coincides with either $\varphi(A)$ or E because $\varphi(A)$ is topologically maximal in E. Suppose $\varphi(A) \cap C = \{0\}$. Since $\varphi(A)C$ and $C\varphi(A)$ are contained in $\varphi(A) \cap C$, it follows that $\varphi(A)C = \{0\} = C\varphi(A)$, so $C \subset ann_E\big(\varphi(A)\big)$, and hence $C = \{0\}$. Thus E has only one nontrivial closed ideal. $\square$

Theorem 6. *Let A be a topological ring with $ann_A(A) = A$, let B be a topologically simple ring with identity, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be a unital ideal extension of A by B. If A is a topologically simple B-bimodule relative to the bimodule structure determined by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$, then E has only one nontrivial closed ideal, namely $\varphi(A)$.*

Proof. Clearly, $\varphi(A)$ is a nonzero closed ideal of E. Moreover, since $E/\varphi(A)$ is topologically isomorphic to B, $\varphi(A)$ is topologically maximal by Lemma 1. Let C be a nonzero closed ideal of E. It is easy to see that $\varphi(A) \cap C$ is then a B-subbimodule of $\varphi(A)$. We cannot have $\varphi(A) \cap C = \{0\}$. For, otherwise it would follow that $\overline{\varphi(A) + C} = E$, since $\overline{\varphi(A) + C}$ would then properly contain $\varphi(A)$. Hence there would exist a net $(a_\lambda)_{\lambda \in L}$ of elements in A and a net $(c_\lambda)_{\lambda \in L}$ of elements in C with $\lim_{\lambda \in L}\big((\varphi(a_\lambda) + c_\lambda\big) = 1$. For any $\lambda, \lambda' \in L$, we would have

$$\big(\varphi(a_\lambda) + c_\lambda\big)\big(\varphi(a_{\lambda'}) + c_{\lambda'}\big) = \varphi(a_\lambda)c_{\lambda'} + c_\lambda \varphi(a_{\lambda'}) + c_\lambda c_{\lambda'} \in C$$

since $\varphi(A)$ has zero multiplication. Taking the limit first relative to λ and then relative to λ', we would obtain that $1 \in C$, so $C = E$, in contradiction with our assumption that $\varphi(A) \cap C = \{0\}$. Thus $\varphi(A) \cap C \neq \{0\}$, and hence $\varphi(A) \cap C = \varphi(A)$ because $\varphi(A)$ is a topologically simple B-bimodule. It follows that $\varphi(A) \subset C$, so that C must coincide with either $\varphi(A)$ or E, because $\varphi(A)$ is topologically maximal in E. Consequently, E has only one nontrivial closed ideal. □

4 Topological rings with only two different nontrivial closed ideals

In this section, we turn our attention to topological rings with exactly two nontrivial closed ideals. First we consider the case when the corresponding ideals are incomparable with respect to inclusion or, equivalently, disjoint.

Theorem 7. *Let E be a topological ring with identity having only two different nontrivial closed ideals. Assume that these ideals are not comparable with respect to inclusion, and let A denote one of them. Then A is a topologically simple ring, $ann_E(A) \neq \{0\}$, E/A is a topologically simple ring with identity, and E can be viewed as a unital ideal extension $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$ of A by E/A, where η is the canonical injection and π is the canonical projection.*

Proof. The assertion follows from Theorem 2 and Lemma 3. □

Theorem 8. *Let A be a topologically simple ring, let B be a topologically simple ring with identity, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be a unital ideal extension of A by B such that $ann_E\big(\varphi(A)\big) \neq \{0\}$. Then E has exactly two nontrivial closed ideals, namely $\varphi(A)$ and $ann_E\big(\varphi(A)\big)$.*

Proof. Clearly, $\varphi(A)$ is a nontrivial closed ideal of E. Moreover, since $E/\varphi(A) \cong B$, $\varphi(A)$ is topologically maximal in E. Further, since E is unital, we must have $ann_E\big(\varphi(A)\big) \neq E$, so $ann_E\big(\varphi(A)\big)$ is a nontrivial closed ideal of E as well.

Let C be an arbitrary nonzero closed ideal of E. Then $\varphi(A) \cap C$ is a closed ideal of $\varphi(A)$. Since $\varphi(A)$ is topologically simple, it follows that either $\varphi(A) \cap C = \{0\}$ or $\varphi(A) \cap C = \varphi(A)$. Assume the former holds. Since $\varphi(A)C$ and $C\varphi(A)$ are contained in $\varphi(A) \cap C$, we conclude that $C \subset ann_E\big(\varphi(A)\big)$. But, since C is nonzero, $\overline{\varphi(A) + C}$ properly contains $\varphi(A)$, so $\overline{\varphi(A) + C} = E$. It follows that

$$\begin{aligned} ann_E\big(\varphi(A)\big) &= E \cdot ann_E\big(\varphi(A)\big) = \overline{\big(\varphi(A) + C\big) \cdot ann_E\big(\varphi(A)\big)} \\ &= \overline{C \cdot ann_E\big(\varphi(A)\big)} \subset C, \end{aligned}$$

and hence $C = ann_E\big(\varphi(A)\big)$.

In the latter case when $\varphi(A) \cap C = \varphi(A)$, we have $\varphi(A) \subset C$. Since $\varphi(A)$ is topologically maximal in E, it follows that C coincides with either $\varphi(A)$ or E. □

In the following, we consider the case of topological rings with exactly two nontrivial closed ideals and such that the corresponding ideals are comparable with respect to inclusion. We first determine under what conditions the topological rings of this type can be realized as ideal extensions of a topologically simple ring by a topological ring with only one nontrivial closed ideal.

Theorem 9. *Let E be a topological ring with identity having only two different nontrivial closed ideals A and B. If $A \subset B$, then E/A is a topological ring with identity containing only one nontrivial closed ideal, and E can be viewed as an ideal extension $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$ of A by E/A, where η is the canonical injection and π is the canonical projection, such that exactly one of the following conditions hold:*

(i) $ann_E(A) = \{0\}$ *and A is a topologically simple ring;*

(ii) $ann_E(A) = A$ *and A, with the structure given by the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, is a topologically simple (E/A)-bimodule;*

(iii) $ann_E(A) = B$, $\overline{B^2}$ *coincides with either A or B, and A, with the structure given by the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, is a topologically simple (E/A)-bimodule;*

(iv) $B^2 = \{0\}$ *and the topological E/B-bimodule B, determined by the sequence $A \xrightarrow{\eta} E \xrightarrow{\pi} E/A$, has only one nontrivial closed subbimodule.*

Proof. Since A and B are the only nontrivial closed ideals of the unital ring E, it follows that $ann_E(A)$ coincides with one of the ideals $\{0\}$, A, or B. Now, if $ann_E(A) = \{0\}$, we must have $ann^l_E(A) = \{0\} = ann^r_E(A)$. For, if one of the ideals $ann^l_E(A)$ or $ann^r_E(A)$ coincided with either A or B, it would follow that $ann_E(A) \neq \{0\}$. Pick an arbitrary nonzero closed ideal C of A, and let $c \in C$ be a nonzero element. It follows from Lemma 7 that $\overline{AcA}$ is a nonzero closed ideal of A and hence of E, so $\overline{AcA} = A$, whence $C = A$. Consequently, A is a topologically simple ring, and hence in this case we are led to (i). Next, if $ann_E(A) = A$, it follows from Lemma 5 that A can be turned into a topological bimodule over E/A. Since every closed subbimodule of A is a closed ideal of E, we deduce that A is a topologically simple E/A-bimodule. Thus in this case we have (ii). Further, assume $ann_E(A) = B$. If $B^2 \neq \{0\}$, it follows from our hypothesis that $\overline{B^2}$ coincides with either A or B. Since, as above, A can be turned into a topologically simple E/A-bimodule, in this case we must have (iii). Finally, if $B^2 = \{0\}$, it follows from Corollary 1 that B can be turned into a topological bimodule over $(E/A)/(B/A) \cong E/B$. Let C be a closed subbimodule of B. We see that for any $x \in E$,

$$xC = \big((x + A) + B/A\big)C \subset C \quad \text{and} \quad Cx = C\big((x + A) + B/A\big) \subset C.$$

It follows that C is a closed ideal of E contained in B, so C must coincides with one of the ideals $\{0\}$, A, or B. Consequently, the topological E/A-bimodule B has only one nontrivial closed subbimodule, and hence in this case we are led to (iv). $\square$

Theorem 10. *Let A be a nonzero topological ring, let B be a topological ring with identity having only one nontrivial closed ideal B_0, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be a unital ideal extension of A by B satisfying one of the following conditions:*

(i) $ann_E\big(\varphi(A)\big) = \{0\}$ *and* A *is a topologically simple ring;*

(ii) $ann_E\big(\varphi(A)\big) = \varphi(A)$ *and* A *is a topologically simple* B*-bimodule relative to the structure given by the sequence* $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;

(iii) $ann_E\big(\varphi(A)\big) = \psi^{-1}(B_0)$, $\overline{\psi^{-1}(B_0)^2}$ *coincides with either* $\varphi(A)$ *or* $\psi^{-1}(B_0)$, *and* A *is a topologically simple* B*-bimodule relative to the structure given by the sequence* $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;

(iv) $\psi^{-1}(B_0)^2 = \{0\}$ *and the topological* (B/B_0)*-bimodule* $\psi^{-1}(B_0)$, *determined by the sequence* $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$, *has only one nontrivial closed subbimodule.*

Then E *has exactly two nontrivial closed ideals, namely* $\varphi(A)$ *and* $\psi^{-1}(B_0)$.

Proof. It is clear that $\varphi(A) \neq \{0\}$ and that $\psi^{-1}(B_0)$ is the only closed ideal of E satisfying $\varphi(A) \subsetneq \psi^{-1}(B_0) \subsetneq E$. Pick an arbitrary closed ideal C of E. If $C \cap \varphi(A) = \varphi(A)$, then $\varphi(A) \subset C$, so that C coincides with one of the ideals $\varphi(A)$, $\psi^{-1}(B_0)$, or E. Assume $C \cap \varphi(A) \neq \varphi(A)$. We shall show that, in any of cases (i)-(iv), $C = \{0\}$. First observe that we must have $C \cap \varphi(A) = \{0\}$. Indeed, this is clear in case (i) holds, since then $C \cap \varphi(A)$ is a closed ideal of the topologically simple ring $\varphi(A)$. Further, in either of cases (ii) or (iii) $C \cap \varphi(A)$ is a closed B-subbimodule of the topologically simple B-bimodule $\varphi(A)$, and so $C \cap \varphi(A) = \{0\}$. Finally, in case (iv) holds, it is clear that $C \cap \varphi(A)$ is a closed (B/B_0)-subbimodule of $\psi^{-1}(B_0)$, so $C \cap \varphi(A) = \{0\}$ because $\psi^{-1}(B_0)$ has only one nontrivial closed subbimodule, namely $\varphi(A)$. This proves that in any of cases (i)-(iv), $C \cap \varphi(A) = \{0\}$. Now, since $C \cdot \varphi(A)$ and $\varphi(A) \cdot C$ are contained in $C \cap \varphi(A)$, it follows that $C \subset ann_E\big(\varphi(A)\big)$. In particular, $C = \{0\}$ if (i) holds. In case (ii) holds, C becomes a closed subbimodule of the topologically simple B-bimodule $\varphi(A)$, so again $C = \{0\}$. Further, in case (iv) holds, we clearly have $ann_E\big(\varphi(A)\big) = \psi^{-1}(B_0)$, so $C = \{0\}$ by our hypothesis that $\varphi(A)$ is the only nontrivial closed (B/B_0)-subbimodule of $\psi^{-1}(B_0)$ and the fact that $C \cap \varphi(A) = \{0\}$. Assume (iii). If we had $C \neq \{0\}$, it would follow that $\overline{C + \varphi(A)} = \psi^{-1}(B_0)$, which would imply

$$\overline{C^2} = \overline{(\overline{C + \varphi(A)})(\overline{C + \varphi(A)})} = \overline{\psi^{-1}(B_0)^2}.$$

But then, in case $\overline{\psi^{-1}(B_0)^2} = \varphi(A)$, we would have $\varphi(A) = \overline{C^2} \subset C \cap \varphi(A)$. Similarly, in case $\overline{\psi^{-1}(B_0)^2} = \psi^{-1}(B_0)$, we would have $C = \psi^{-1}(B_0)$. In both cases the derived conclusion is in contradiction with the fact that $C \cap \varphi(A) = \{0\}$. ☐

Next we complete the picture by determining under what conditions topological rings with exactly two nontrivial closed ideals can be realized as extensions of a topological ring with only one nontrivial closed ideal by a topologically simple ring.

Definition 9. Let E be a topological ring. A closed ideal M of E is said to be a topologically minimal ideal of E if $M \neq \{0\}$ and for every closed ideal C of E such that $C \subset M$, either $C = \{0\}$ or $C = M$.

Theorem 11. *Let E be a topological ring with identity having only two different nontrivial closed ideals A and B. If $A \subset B$, then E/B is a topologically simple ring with identity, and E can be viewed as an ideal extension $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$ of B by E/B, where η is the canonical injection and π is the canonical projection, such that exactly one of the following conditions hold:*

(i) *The ideals $\overline{AB}$, $\overline{BA}$ and $ann_E(B/A)$ coincide with A, and B has only one nontrivial closed ideal;*

(ii) *$\overline{AB} = A = \overline{BA}$, $ann_E(B/A) = B$, A is a topologically minimal ideal of B, and the topological E/B-bimodule B/A, determined by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$, is topologically simple;*

(iii) *$\overline{AB} = A = ann^r_E(B)$, $B^2 = B$, and A is a topologically maximal ideal of B and a unital topologically simple left E/B-module relative to the structure given by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$;*

(iv) *$\overline{BA} = A = ann^l_E(B)$, $B^2 = B$, and A is a topologically maximal ideal of B and a unital topologically simple right E/B-module relative to the structure given by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$;*

(v) *$ann_E(B) = A$, $ann_E(B/A) = A$, $\overline{B^2} = B$, and A is a topologically maximal ideal of B and a unital topologically simple E/B-bimodule relative to the structure given by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$;*

(vi) *$ann_E(B) = A$, $ann_E(B/A) = B$, and A and B/A are unital topologically simple E/B-bimodules relative to the structures given by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$;*

(vii) *$ann_E(B) = B$, and the topological E/B-bimodule B, determined by the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$, has only one nontrivial closed subbimodule.*

Proof. Since A and B are the only nontrivial closed ideals of E and since $A \subset B$, it is clear that E/B is a topologically simple ring. It is also clear that $ann_E(B)$ coincides with one of the ideals $\{0\}$, A, or B.

We first consider the case when $ann_E(B) = \{0\}$. Then, clearly, at least one of the ideals $\overline{AB}$ and $\overline{BA}$ is nonzero. Suppose first that $\overline{AB}$ and $\overline{BA}$ are both nonzero. Since $\overline{AB}$ and $\overline{BA}$ are contained in A, it follows that $\overline{AB} = A = \overline{BA}$. In particular, since A is the smallest nonzero closed ideal of E, we conclude that $ann^l_E(B) = \{0\} = ann^r_E(B)$. Further, since $A \subset ann_E(B/A)$, we have either $ann_E(B/A) = A$ or $ann_E(B/A) = B$. Assume the former holds. Then we must have $ann^l_E(B/A) = A$ and $ann^r_E(B/A) = A$. For, if we had either $ann^l_E(B/A) = B$ or $ann^r_E(B/A) = B$, it would follow that $ann_E(B/A) = B$, a contradiction. Thus $ann^l_E(B/A) = A = ann^r_E(B/A)$. Pick an arbitrary nonzero closed ideal C of B. Given any nonzero $c \in C$, it follows from Lemma 7 that $\overline{BcB}$ is a nonzero ideal of B and hence of E, whence $\overline{BcB}$ coincides with either A or B. Consequently, if $C \subset A$, we must have $C = A$.

Suppose $C \not\subset A$, and pick any $c \in C \setminus A$. Since $ann^l_E(B/A) = A$, there exists $b \in B$ such that $cb \notin A$. Similarly, since $ann^r_E(B/A) = A$, there exists $b' \in B$ such that $b'cb \notin A$. It follows that $\overline{BcB} = B$, so $C = B$, and hence in this case we are led to (i). Now assume the latter case when $ann_E(B/A) = B$ holds. Then $(B/A)^2 = \{0\}$, so that, by Corollary 1, B/A can be turned into a topological bimodule over E/B by setting

$$(b + A)(x + B) = (b + A)(x + A) = bx + A$$

and

$$(x + B)(b + A) = (x + A)(b + A) = xb + A$$

for all $b \in B$ and $x \in E$. To see that this bimodule is topologically simple, pick an arbitrary closed E/B-subbimodule C' of B/A. Letting $\varphi : B \to B/A$ be the canonical projection, set $C = \varphi^{-1}(C')$. Since, for any $c \in C$ and $x \in E$, we have $cx + A = (c + A)(x + B) \in C'$ and $xc + A = (x + B)(c + A) \in C'$, it follows that C is a proper closed ideal of E containing A, so C coincides with either A or B, which proves that C' is trivial in B/A. Further, given any nonzero $a \in A$, we deduce by Lemma 7 and the fact that $B^2 \subset A$, that $\overline{BaB}$ is a nonzero closed ideal of B and hence of E, which is contained in A, whence $\overline{BaB} = A$. It follows that A is a topologically minimal ideal of B, so in this case we have (ii).

Now let us suppose that $\overline{AB} \neq \{0\}$ and $\overline{BA} = \{0\}$. Then, clearly, $\overline{AB} = A$ and $\overline{B^2} \neq \{0\}$. If we had $\overline{B^2} = A$, it would follow that $\overline{AB} = \overline{\overline{B^2}B} = \overline{B\overline{B^2}} = \overline{BA} = \{0\}$, a contradiction. Thus $\overline{B^2} = B$, and hence $ann^r_E(B) = A$. By using the sequence $B \xrightarrow{\eta} E \xrightarrow{\pi} E/B$, we see from Lemma 6 that A can be turned into a topological left E/B-module. If C is a closed submodule of A, then C is clearly a closed ideal of E contained in A, so either $C = \{0\}$ or $C = A$. This proves that A is a topologically simple E/B-module. Now let C be a closed ideal of B properly containing A, and pick any $c \in C \setminus A$. Since $\overline{B^2} = B$, there is $b \in B$ such that $bc \notin A$. Analogously, there is $b' \in B$ such that $bcb' \notin A$. It follows that $\overline{BcB}$ is a closed ideal of B, and hence of E, which properly contains A, so $\overline{BcB} = B$, whence $C = B$. Consequently, A is topologically maximal in B, and thus in this case we have (iii).

Similarly, in the remaining case when $\overline{AB} = \{0\}$ and $\overline{BA} \neq \{0\}$, we have (iv).

Next we consider the case when $ann_E(B) = A$. It follows from Lemma 5 that A can be turned into a topological bimodule over E/B by setting $a(x + B) = ax$ and $(x + B)a = xa$ for all $a \in A$ and $x \in E$. Letting C be a nonzero closed E/B-subbimodule of A, pick any $c \in C$ and $x \in E$. Since $cx = c(x + B) \in C$ and $xc = (x + B)c \in C$, we see that C is an ideal of E, which gives $C = A$. Hence A is a topologically simple E/B-bimodule. Further, let us consider $ann_E(B/A)$. We must have either $ann_E(B/A) = A$ or $ann_E(B/A) = B$. If the former holds, then $B^2 \not\subset A$, so that $\overline{B^2} = B$. We also deduce as above that $ann^l_E(B/A) = A = ann^r_E(B/A)$. Let C be an arbitrary closed ideal of B properly containing A, and pick any $c \in C \setminus A$. Since $ann^l_E(B/A) = A$, there exists $b \in B$ such that $cb \notin A$. Similarly, since $ann^r_E(B/A) = A$, there exists $b' \in B$ such that $b'cb \notin A$. It follows that $\overline{BcB}$ is a closed ideal of E which is not contained in A, so $\overline{BcB} = B$, whence $C = B$, proving that A is topologically maximal in B. Hence in this case we are led to (v). Now assume the latter case when $ann_E(B/A) = B$ holds. Then $(B/A)^2 = \{0\}$, so B/A can be turned into a topological bimodule over E/B. As above, we can see that if C' is arbitrary nonzero closed E/B-subbimodule of B/A and $\varphi : B \to B/A$ is the canonical projection, then $\varphi^{-1}(C')$ coincides with either A or B. Consequently, in this case we are led to (vi).

Now we consider the case when $ann_E(B) = B$. By Lemma 5, B can be turned into a topological bimodule over E/B. If C is a nontrivial closed E/B-subbimodule of B, it is easy to see that C is an ideal of E, and so we must have $C = A$. Thus in this case we are led to (vii). ☐

Theorem 12. *Let A be a topological ring having a nontrivial closed ideal A_0, let B be a topologically simple ring with identity, and let $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$ be a unital ideal extension of A by B satisfying one of the following conditions:*

(i) $\overline{A_0A} = A_0 = \overline{AA_0}$, $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A_0)$, *and A has only one nontrivial closed ideal;*

(ii) $\overline{A_0A} = A_0 = \overline{AA_0}$, $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A)$, *$A_0$ is a topologically minimal ideal of A, and the topological B-bimodule A/A_0, determined by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$, is topologically simple;*

(iii) $\varphi(\overline{A_0A}) = \varphi(A_0) = ann^r_E\big(\varphi(A)\big)$, $A^2 = A$, *and A_0 is a topologically maximal ideal of A and a unital topologically simple left B-module relative to the structure given by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;*

(iv) $\varphi(\overline{AA_0}) = \varphi(A_0) = ann^l_E\big(\varphi(A)\big)$, $A^2 = A$, *and A_0 is a topologically maximal ideal of A and a unital topologically simple right B-module relative to the structure given by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;*

(v) $ann_E\big(\varphi(A)\big) = \varphi(A_0)$, $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A_0)$, $\overline{A^2} = A$, *and A_0 is a topologically maximal ideal of A and a unital topologically simple B-bimodule relative to the structure given by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;*

(vi) $ann_E\big(\varphi(A)\big) = \varphi(A_0)$, $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A)$, *and A_0 and A/A_0 are unital topologically simple B-bimodules relative to the structures given by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$;*

(vii) $ann_E\big(\varphi(A)\big) = \varphi(A)$, *and the topological B-bimodule A, determined by the sequence $A \xrightarrow{\varphi} E \xrightarrow{\psi} B$, has only one nontrivial closed subbimodule.*

Then E has exactly two nontrivial closed ideals, namely $\varphi(A_0)$ and $\varphi(A)$.

Proof. Clearly, $\varphi(A_0)$ and $\varphi(A)$ are distinct nontrivial closed ideals of E satisfying $\varphi(A_0) \subset \varphi(A)$. Moreover, $\varphi(A)$ is topologically maximal in E because $E/\varphi(A) \cong B$. Let C be an arbitrary closed ideal of E. If $C \cap \varphi(A) = \varphi(A)$, then $\varphi(A) \subset C$, so that C coincides with one of the ideals $\varphi(A)$ or E since $\varphi(A)$ is topologically maximal.

Assume $C \cap \varphi(A) \neq \varphi(A)$. We first show that in any of cases (i) - (vii), $C \cap \varphi(A)$ coincides with either $\{0\}$ or $\varphi(A_0)$. Indeed, this is clear in case (i) holds because $C \cap \varphi(A)$ is a closed ideal of $\varphi(A)$.

Assume (ii) holds, and suppose $C \cap \varphi(A) \neq \{0\}$. Since $ann_E\big(\varphi(A)\big) = \{0\}$, we cannot have $\big(C \cap \varphi(A)\big)\varphi(A) = \{0\} = \varphi(A)\big(C \cap \varphi(A)\big)$. On the other hand, $\big(C \cap \varphi(A)\big)\varphi(A)$ and $\varphi(A)\big(C \cap \varphi(A)\big)$ are contained in $\varphi(A_0)$ by our hypothesis that $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A)$. Since $\varphi(A_0)$ is topologically minimal in $\varphi(A)$, it follows that either $\overline{\big(C \cap \varphi(A)\big)\varphi(A)}$ or $\overline{\varphi(A)\big(C \cap \varphi(A)\big)}$ coincides with $\varphi(A_0)$, whence $\varphi(A_0) \subset C \cap \varphi(A)$. As the B-bimodule $\varphi(A)/\varphi(A_0)$ is topologically simple, we deduce that $C \cap \varphi(A)$ coincides with $\varphi(A_0)$.

In the following, we consider (iii), (iv), (v) and (vi) simultaneously. By hypotheses, in every of cases (iii), (iv) and (v) we have $\overline{\varphi(A)^2} = \varphi(A)$. We first show that if (vi) holds, then $\overline{\varphi(A)^2} = \varphi(A_0)$. Indeed, since $ann_E\big(\varphi(A)/\varphi(A_0)\big) = \varphi(A)$, we have $\overline{\varphi(A)^2} \subset \varphi(A_0)$, so that $\overline{\varphi(A)^2}$ is a closed B-subbimodule of $\varphi(A_0)$. Moreover, $\overline{\varphi(A)^2} \neq \{0\}$ because $ann_E\big(\varphi(A)\big) = \varphi(A_0)$. Since $\varphi(A_0)$ is a topologically simple B-bimodule, we get $\overline{\varphi(A)^2} = \varphi(A_0)$.

Now, in every of cases (iii), (iv) and (v), if $C \cap \varphi(A) \subset \varphi(A_0)$, we must have either $C \cap \varphi(A) = \{0\}$ or $C \cap \varphi(A) = \varphi(A_0)$ because $C \cap \varphi(A)$ is a B-submodule (respectively, B-subbimodule) of $\varphi(A_0)$ and $\varphi(A_0)$ is topologically simple. We next show that $C \cap \varphi(A) \not\subset \varphi(A_0)$ leads to a contradiction. Indeed, suppose $C \cap \varphi(A) \not\subset \varphi(A_0)$, so that $\overline{(C \cap \varphi(A)) + \varphi(A_0)}$ properly contains $\varphi(A_0)$. Consequently, in every of cases (iii), (iv) and (v), we have $\overline{(C \cap \varphi(A)) + \varphi(A_0)} = \varphi(A)$ because $\varphi(A_0)$ is topologically maximal in $\varphi(A)$. Further, in case (vi) holds, it is easy to see that $\overline{(C \cap \varphi(A)) + \varphi(A_0)}/\varphi(A_0)$ is a nonzero closed B-subbimodule of $\varphi(A)/\varphi(A_0)$. Since $\varphi(A)/\varphi(A_0)$ is topologically simple, it follows that $\overline{(C \cap \varphi(A)) + \varphi(A_0)}/\varphi(A_0) = \varphi(A)/\varphi(A_0)$, so again $\overline{(C \cap \varphi(A)) + \varphi(A_0)} = \varphi(A)$. We then have

$$\overline{\varphi(A)^2} = \overline{(\overline{(C \cap \varphi(A)) + \varphi(A_0)})\varphi(A)} = \overline{(C \cap \varphi(A))\varphi(A)} \subset C \cap \varphi(A).$$

Therefore either of equalities $\overline{\varphi(A)^2} = \varphi(A)$ or $\overline{\varphi(A)^2} = \varphi(A_0)$ together with the fact that $\overline{(C \cap \varphi(A)) + \varphi(A_0)} = \varphi(A)$ gives $C \cap \varphi(A) = \varphi(A)$, a contradiction.

Finally, if (vii) holds, then clearly $C \cap \varphi(A)$ is a B-subbimodule of $\varphi(A)$, and hence $C \cap \varphi(A)$ must coincide with either $\{0\}$ or $\varphi(A_0)$.

Thus, in any of cases (i)-(vii), $C \cap \varphi(A)$ coincides with either $\{0\}$ or $\varphi(A_0)$. Now, since $C\varphi(A)$ and $\varphi(A)C$ are contained in $C \cap \varphi(A)$, we have $C \subset ann_E\big(\varphi(A)\big)$ if $C \cap \varphi(A) = \{0\}$ and $C \subset ann_E\big(\varphi(A)/\varphi(A_0)\big)$ if $C \cap \varphi(A) = \varphi(A_0)$. It follows that if $C \cap \varphi(A) = \{0\}$, then $C = \{0\}$ in any of cases (i)-(vii). Similarly, if $C \cap \varphi(A) = \varphi(A_0)$, then $C = \varphi(A_0)$ in any of cases (i)-(vii). $\square$

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