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An Approach for Determining the Matrix of Limiting State Probabilities in Discrete Markov Processes

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Abstract. A new approach for determining the matrix of limiting state probabilities in Markov processes is proposed and a polynomial time algorithm for calculating this matrix is grounded. The computational complexity of the algorithm is $O(n^4)$, where n is the number of the states of the discrete system.

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1 Introduction and Problem Formulation

Consider a stochastic discrete system L with finite set of states

$$X = \{x_1,\ x_2,\ \ldots,\ x_n\}.$$

Assume that the dynamics of the system is modeled by a Markov process with given stochastic matrix of probabilities transitions $P = (p_{ij})_{i,j=\overline{1,n}}$ where

$$\sum_{j=1}^{n} p_{i,j} = 1,\ i = \overline{1,n}; \quad 0 \le p_{i,j} \le 1,\ i,j = \overline{1,n}.$$

The probability $P_{x_{i_0}}(x,t)$ of system's passage from the state x_{i_0} to an arbitrary state $x \in X$ by using t transitions is defined and calculated on the basis of the following recursive formula [2]

$$P_{x_{i_0}}(x,\tau+1) = \sum_{y \in X} P_{x_{i_0}}(y,\tau) p_{y,x}, \quad \tau = \overline{0,t-1}, \tag{1}$$

where $P_{x_{i_0}}(x_{i_0},0) = 1$ and $P_{x_{i_0}}(x,0) = 0,\ \forall x \in X \setminus \{x_{i_0}\}$. We call these probabilities

state-time probabilities of system L. Formula (1) can be represented in the matrix form as follow

$$\pi(\tau+1) = \pi(\tau)P, \quad \tau = \overline{0,t-1}. \tag{2}$$

Here $\pi(\tau) = (\pi_1(\tau), \pi_2(\tau), \ldots, \pi_n(\tau))$ is the vector, where an arbitrary component i expresses the probability of the system L to reach the state x_i from x_{i_0} at the moment of time τ, i.e. $\pi_i(\tau) = P_{x_{i_0}}(x_i, \tau)$. At the starting moment of time $\tau = 0$ the vector $\pi(\tau)$ is given and its components are defined as follows: $\pi_{i_0}(0) = 1$ and $\pi_i(0) = 0$ for arbitrary $i \neq i_0$. It is easy to observe that if for given starting vector $\pi(0)$ we apply formula (2) for $\tau = 0, 1, 2, \ldots, t-1$, then we obtain

$$\pi(t) = \pi(0)P^t,$$

where $P^t = P \times P \times \cdots \times P$. So, an arbitrary element $p^{(t)}_{x_i,x_j}$ of this matrix expresses the probability of system L to reach the state x_j from x_i by using t units of times. It is easy to see that for given starting representation of the vector $\pi(0)$ the following properties holds

$$\sum_{i=1}^{n} \pi_i(\tau) = 1, \quad \tau = 0, 1, 2, \ldots. \tag{3}$$

The correctness of this property can be easy proved using induction principle with respect to τ. Indeed, for $\tau = 0$ the equality (3) holds according to the definition. If we assume that (3) holds for every $\tau \leq t$ then we obtain the correctness of this formula for $\tau = t + 1$ as follows

$$\sum_{i=1}^{n} \pi_i(t+1) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{x_j,x_i} \pi_j(t) = \sum_{j=1}^{n} \pi_j(t) \sum_{i=1}^{n} p_{x_j,x_i} = \sum_{j=1}^{n} \pi_j(t) = 1.$$

So, formula (3) holds. In order to analyze the asymptotic behavior of the state-time probabilities of the system using formula (3) we will assume that there exists the limit

$$\lim_{t\to\infty} P^t = Q.$$

If this limit exists then there exists the following limit

$$\pi = \lim_{t\to\infty} \pi(t) = \pi(0) \lim_{t\to\infty} P^t = \pi(0)Q,$$

where an arbitrary component π_j of the vector $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ expresses the probability that the system L will occupy the state x_j after a large number of transitions when it starts transitions in the state x_{i_0}. The vector π will be called the vector of limiting state probabilities. Based on the mentioned above property we may conclude that

$$\sum_{j=1}^{n} \pi_j = 1$$

for an arbitrary given starting vector $\pi(0)$. This means that the matrix $Q = (q_{x,y})$ satisfies the condition

$$\sum_{y\in X} q_{x,y} = 1, \quad \forall x \in X,$$

where $q_{x,y} \geq 0, \quad \forall x, y \in X$, i.e. $Q = (q_{x,y})$ is a stochastic matrix. An arbitrary element $q_{x,y}$ of this matrix expresses the probability that the system will occupy the state y after a large number of transitions if it starts transitions in the state x. The matrix Q is called the matrix of limiting states probabilities of the Markov process.

An important class of discrete Markov process represents ergodic Markov chain. For this class all rows of the matrix of limiting states probabilities Q are the same, i.e. $q_{x,y} = q_{v,y}, \quad \forall x, y, v \in X$. In this case the limiting state probabilities $\pi_j,\ j = \overline{1,n}$, does not depend on the state in which the system starts transitions. The vector π of limiting state probabilities can be found by solving the system of linear equations

$$\begin{cases} \pi = \pi P \\ \sum\limits_{j=1}^{n} \pi_j = 1. \end{cases} \tag{4}$$

The first condition $\pi = \pi P$ in this system is obtained from (2) when $\tau \to \infty$ and the second one reflects the property that after a large number of transitions the dynamical system will be in one of the states $x_j \in X$. It is well known that for ergodic Markov chains the system (4) has a unique solution [2, 4]. The necessary and sufficient conditions for the ergodicity of Markov processes are given in [2, 4]. In general system (4) may have a unique solution also when the limit $\lim\limits_{t\to\infty} P^t$ does not exist. This case may correspond to periodic Markov process and a component π_j of vector π that satisfies (4) can be treated as the probability of the system L to occupy the state x_j at the random moment of times during a large number of transitions. In the following we can see that the definition of the matrix of limiting-state probabilities Q can be extended for an arbitrary Markov process, however in the case when $\lim\limits_{t\to\infty} P^t$ does not exist the elements of the matrix Q have another interpretation.

In this paper we describe an approach for determining the matrix of limiting state probabilities in Markov processes and propose a polynomial time algorithm for calculating of this matrix. We show that the running time the algorithm is $O(n^4)$, where n is the number of the states of the discrete system.

2 The main results

The aim of this section is to ground a polynomial time algorithm for determining the limit matrix Q for an arbitrary discrete Markov process with given stochastic matrix P. We describe such an algorithm which is based on the idea of z-transform and classical numerical methods.

2.1 The Main Approach and the General Scheme of the Algorithm

Let $\mathbb{C}$ be the complex space and denote by $\mathbb{M}(\mathbb{C})$ the set of complex matrices with n rows and n columns. We consider the function $A : \mathbb{C} \to \mathbb{M}(\mathbb{C})$, where

$$A(z) = I - zP, \quad z \in \mathbb{C}.$$

We denote the elements of the matrix $A(z)$ by $a_{i,j}(z),\ i, j = \overline{1,n}$, i.e.

$$a_{i,j}(z) = \delta_{i,j} - zp_{ij} \in \mathbb{C}[z]$$

where

$$\delta_{i,j} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases} \quad i,j = \overline{1,n}.$$

It is evident that the determinant $\Delta(z)$ of the matrix $A(z)$ is a polynomial of degree less or equal to n, $(deg(\Delta(z)) \leq n,\ \Delta(z) \in \mathbb{C}[z])$. Therefore if we denote $\mathcal{D} = \{z \in \mathbb{C} \mid \Delta_P(z) \neq 0\}$ then we obtain that $|\mathbb{C}\backslash\mathcal{D}| \leq deg(\Delta(z)) \leq n$ and for an arbitrary $z \in \mathcal{D}$ there exists the inverse matrix of $A(z)$. So, we can define the function $F : \mathcal{D} \to \mathbb{M}(\mathbb{C})$ where

$$F(z) = (A(z))^{-1}.$$

Then the elements $F_{i,j}(z),\ i,j = \overline{1,n}$ of $F(z)$ can be found as follows:

$$F_{i,j}(z) = \frac{M_{j,i}(z)}{\Delta(z)},\ i,j = \overline{1,n},$$

where

$$M_{i,j}(z) = (-1)^{i+j} A_{i,j}(z)$$

and $A_{i,j}(z)$ is the determinant of the matrix obtained from $A(z)$ by deleting the row i and the column j, $i,j = \overline{1,n}$. Therefore

$$M_{j,i}(z) \in \mathbb{C}[z],\ deg(M_{j,i}(z)) \leq n-1,\ i,j = \overline{1,n}.$$

Note that $\Delta(1) = 0$ because for the matrix $A(1)$ holds the property

$$\sum_{j=1}^{n}(\delta_{ij} - p_{ij}) = \sum_{j=1}^{n}\delta_{ij} - \sum_{j=1}^{n}p_{ij} = \delta_{ii} - 1 = 0,\ i = \overline{1,n}.$$

This means that $1 \in \mathbb{C}\backslash\mathcal{D}$ and therefore $\Delta(z)$ can be factored by $(z-1)$. Taking into account that $F_{i,j}(z)$ is a rational fraction with the denominator $\Delta(z)$ we can represent $F_{i,j}(z)$ uniquely in the following form

$$F_{i,j}(z) = B_{ij}(z) + \sum_{y \in \mathbb{C}\backslash\mathcal{D}_P} \sum_{k=1}^{m(y)} \frac{\alpha_{i,j,k}(y)}{(z-y)^k},\ i,j = \overline{1,n}, \tag{5}$$

where $m(z)$ is the order of the root z of the polynomial $\Delta(z)$, $z \in \mathbb{C}\backslash\mathcal{D}$, and $\alpha_{ijk}(y) \in \mathbb{C},\ \forall y \in \mathbb{C}\backslash\mathcal{D},\ k = \overline{1,m(y)},\ i,j = \overline{1,n}$. In this representation of $F_{i,j}(z)$ the degree of the polynomial $B_{ij}(z) \in \mathbb{C}[z]$ satisfies the condition

$$deg(B_{i,j}(z)) = deg(M_{j,i}(z)) - deg(\Delta(z)),$$

where $deg(M_{j,i}(z)) \geq deg(\Delta(z))$, otherwise $B_{i,j}(z) = 0$.

To represent (5) in a more convenient form we shall use some elementary properties of the function $\nu_k(z) = \dfrac{1}{(1-z)^k}$, $k = 1,2,\ldots$. It is well known that in the case $k = 1$ the function $\nu_1(z)$ admits the series expansion $\nu_1(z) = \sum_{t=0}^{\infty} z^t$. In general case (for an arbitrary $k > 1$) the following recursive relation holds $\nu_{k+1}(z) = \dfrac{d\nu_k(z)}{kdz}$, $k = 1, 2, \ldots$. Using these properties and induction principle we can obtain the series expansion of the function $\nu_k(z)$, $\forall k \geq 1$: $\nu_k(z) = \sum_{t=0}^{\infty} T_{k-1}(t) z^t$, where $T_{k-1}(t)$ is a polynomial of degree less or equal to $(k-1)$.

Based on mentioned above properties we can make the following transformation in (5) we can make the following transformation:

$$F_{i,j}(z) = B_{i,j}(z) + \sum_{y \in \mathbb{C} \backslash \mathcal{D}} \sum_{k=1}^{m(y)} \frac{\left(-\frac{1}{y}\right)^k \alpha_{i,j,k}(y)}{\left(1 - \frac{1}{y} z\right)^k} =$$

$$= B_{i,j}(z) + \sum_{y \in \mathbb{C} \backslash \mathcal{D}} \sum_{k=1}^{m(y)} \left(-\frac{1}{y}\right)^k \alpha_{i,j,k}(y) \nu_k \left(\frac{z}{y}\right) =$$

$$= B_{i,j}(z) + \sum_{y \in \mathbb{C} \backslash \mathcal{D}} \sum_{k=1}^{m(y)} \left(-\frac{1}{y}\right)^k \alpha_{i,j,k}(y) \sum_{t=0}^{\infty} T_{k-1}(t) \left(\frac{z}{y}\right)^t =$$

$$= B_{i,j}(z) + \sum_{t=0}^{\infty} \left(\frac{z}{y}\right)^t \sum_{y \in \mathbb{C} \backslash \mathcal{D}} \sum_{k=0}^{m(y)-1} \left(-\frac{1}{y}\right)^{k+1} \alpha_{i,j,k+1}(y) T_k(t).$$

We can observe that in the relation above the expression

$$\sum_{k=0}^{m(y)-1} \left(-\frac{1}{y}\right)^{k+1} \alpha_{i,j,k+1}(y) T_k(t)$$

represents a polynomial of degree less or equal to $m(y) - 1$ and we can write it in the form $\sum_{k=0}^{m(y)-1} \beta_{i,j,k}(y) t^k$, where $\beta_{i,j,k}$ represent the corresponding coefficients of this $\sum_{k=0}^{m(y)-1} \beta_{i,j,k}(y) t^k$ for polynomial. Therefore if in the expression above we substitute $\sum_{k=0}^{m(y)-1} \left(-\frac{1}{y}\right)^{k+1} \alpha_{i,j,k+1}(y) T_k(t)$ then we obtain

$$F_{i,j}(z) = B_{i,j}(z) + \sum_{t=0}^{\infty} z^t \sum_{y \in \mathbb{C} \backslash \mathcal{D}} \sum_{k=0}^{m(y)-1} \frac{t^k}{y^t} \beta_{i,j,k}(y) =$$

$$= W_{i,j}(z) + \sum_{t=1+deg(B_{i,j}(z))}^{\infty} z^t \sum_{y\in\mathbb{C}\setminus\mathcal{D}} \sum_{k=0}^{m(y)-1} \frac{t^k}{y^t}\beta_{i,j,k}(y),\ i,j=\overline{1,n}, \tag{6}$$

where $\beta_{i,j,k}(y) \in \mathbb{C},\ \forall y \in \mathbb{C}\setminus\mathcal{D},\ k = \overline{0, m(y)-1},\ i,j = \overline{1,n}$, and $W_{ij}(z) \in \mathbb{C}[z]$ is a polynomial of degree that satisfies the condition $deg(W_{i,j}(z)) = deg(B_{i,j}(z))$, $i,j = \overline{1,n}$.

Note that for the norm of the matrix P we have $\|P\| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} p_{i,j} = 1$, and therefore $\|zP\| = |z|\|P\| = |z|$. Let $|z| < 1$. Then for $F(z)$ we have

$$F(z) = (I - zP)^{-1} = \sum_{t=0}^{\infty} P^t z^t.$$

This means that

$$F_{i,j}(z) = \sum_{t=0}^{\infty} p_{i,j}(t) z^t,\ i,j = \overline{1,n}. \tag{7}$$

From definition of z-transform and from (6) – (7) we obtain

$$p_{i,j}(t) = \sum_{y\in\mathbb{C}\setminus\mathcal{D}} \sum_{k=0}^{m(y)-1} \frac{t^k}{y^t}\beta_{i,j,k}(y),\ \forall t > deg(B_{i,j}(z)),\ i,j = \overline{1,n}.$$

Since $0 \le p_{i,j}(t) \le 1,\ i,j = \overline{1,n},\ \forall t \ge 0$, we have

$$|y| \ge 1,\ \forall y \in \mathbb{C}\setminus\mathcal{D},\ \beta_{i,j,k}(1) = 0,\ \forall k \ge 1.$$

This involves $\alpha_{i,j,k}(1) = 0,\ \forall k \ge 2$.

Now let us assume that $\Delta(z) = (z-1)^{m(1)} T(z),\ T(1) \ne 0$. Then the relation (5) is represented as follows:

$$F_{i,j}(z) = \frac{\alpha_{i,j,1}(1)}{z-1} + B_{i,j}(z) + \sum_{y\in(\mathbb{C}\setminus\mathcal{D})\setminus\{1\}} \sum_{k=1}^{m(y)} \frac{\alpha_{i,j,k}(y)}{(z-y)^k} =$$

$$= \frac{\alpha_{i,j,1}(1)}{z-1} + \frac{Y_{i,j}(z)}{T(z)},\ i,j = \overline{1,n},$$

where $Y_{i,j}(z) \in \mathbb{C}[z]$ and

$$deg(Y_{i,j}(z)) = deg(B_{i,j}(z)) + deg(T(z)) = deg(B_{i,j}(z)) + deg(\Delta(z)) - m(1) =$$

$$= deg(M_{j,i}(z)) - m(1) \le n - 1 - m(1) \le n - 2,\ i,j = \overline{1,n}.$$

In the following we will denote

$$Y(z) = (Y_{i,j}(z))_{i,j=\overline{1,n}},\ \alpha_1(1) = (\alpha_{i,j,1}(1))_{i,j=\overline{1,n}}.$$

Then the matrix $F(z)$ can be represented as follows:

$$F(z) = \frac{1}{z-1}\,\alpha_1(1) + \frac{1}{T(z)}\,Y(z). \tag{8}$$

From this formula and from definition of the limiting-state matrix Q we have

$$Q = -\alpha_1(1), \tag{9}$$

i.e Q in the inverse matrix of $(I - zP)$ corresponds to the term with the coefficient $\dfrac{1}{1-z}$.

From (8) and (9) we obtain formula

$$Q = \lim_{z\to 1}(1-z)(I-zP)^{-1}.$$

In the following we show how to determine the polynomial $\Delta(z)$ and the function $F(z)$ in the matrix form.

2.2 Algorithm for Determining the Polynomial $\Delta(z)$

Let us consider the characteristic polynomial

$$K(z) = |P - zI| = \sum_{k=0}^{n} \nu_k z^k.$$

In this polynomial the coefficient of the term with maximal degree of variable z is $\nu_n = |-I_n| = (-1)^n \neq 0$. This means that $deg(K(z)) = n$ and we can represent $K(z)$ in the form

$$K(z) = (-1)^n(z^n - \alpha_1 z^{n-1} - \alpha_2 z^{n-2} - \ldots - \alpha_n).$$

If we denote $\alpha_0 = -1$, then it is easy to see that the coefficients ν_k can be represented as follows:

$$\nu_k = (-1)^{n+1}\alpha_{n-k},\ k = \overline{0,n}.$$

In $[1,5]$ it is shown that the coefficients α_k can be calculated basing on Leverrier's method using $O(n^3)$ elementary operations. This method can be applied for determining the coefficients α_k in the following way:

1) We determine the matrices

$$P^{(k)} = (p_{i,j}(k))_{i,j=\overline{1,n}},\ k = \overline{1,n},$$

where $P^{(k)} = P \times P \times \cdots \times P$;

2) Then we determine the traces of these matrices:

$$s_k = trP^{(k)} = \sum_{j=1}^{n} p_{j,j}(k), \ k = \overline{1,n};$$

3) Finally we calculate the coefficients

$$\alpha_k = \frac{1}{k}\left(s_k - \sum_{j=1}^{k-1} \alpha_j s_{k-j}\right), \ k = \overline{1,n}.$$

If the coefficients α_k are known then we can determine the coefficients of the polynomial $\Delta(z) = \sum\limits_{k=0}^{n} \beta_k z^k$. Indeed, if $z \in \mathbb{C}\backslash\{0\}$ then

$$\Delta(z) = |I - zP| = (-z)^n \left|P - \frac{1}{z}I\right| = (-1)^n z^n K\left(\frac{1}{z}\right) =$$

$$= (-1)^n z^n \sum_{k=0}^{n} \nu_k \frac{1}{z^k} = (-1)^n \sum_{k=0}^{n} \nu_k z^{n-k} = \sum_{k=0}^{n} (-1)^n \nu_{n-k} z^k =$$

$$= \sum_{k=0}^{n} (-1)^n (-1)^{n+1} \alpha_k z^k = \sum_{k=0}^{n} (-\alpha_k) z^k.$$

For $z = 0$ we have

$$\Delta(0) = |I| = 1 = -\alpha_0.$$

Therefore finally we obtain

$$\Delta(z) = \sum_{k=0}^{n} (-\alpha_k) z^k, \ \forall z \in \mathbb{C}.$$

This means $\beta_k = -\alpha_k, \ k = \overline{0,n}$. So, the coefficients $\beta_k, \ k = \overline{0,n}$, can be calculated using a similar recursive formula

$$\beta_k = -\alpha_k = -\frac{1}{k}\left(s_k - \sum_{j=1}^{k-1} \alpha_j s_{k-j}\right) = -\frac{1}{k}\left(s_k + \sum_{j=1}^{k-1} \beta_j s_{k-j}\right), \ k = \overline{1,n},$$

$$\beta_0 = -\alpha_0 = 1,$$

We can use the following algorithm for determining the coefficients β_k.

Algorithm 1.1: Determining the coefficients of the polynomial $\boldsymbol{\Delta(z)}$

1) Calculate the matrices $P^{(k)} = (p_{i,j}(k))_{i,j=\overline{1,n}},\ k = \overline{1,n}$;

2) Determine the traces of the matrices $P^{(k)}$:

$$s_k = trP^{(k)} = \sum_{j=1}^{n} p_{j,j}(k),\ k = \overline{1,n};$$

3) Find the coefficients

$$\beta_0 = 1,\ \beta_k = -\frac{1}{k}\left(s_k + \sum_{j=1}^{k-1} \beta_j s_{k-j}\right),\ k = \overline{1,n}.$$

2.3 Polynomial Time Algorithm for Determining the Function $\boldsymbol{F(z)}$

Consider
$$T'(z) = (z-1)T(z)$$
and denote $N = deg(T'(z)) = n - (m(1) - 1)$. We have already shown that the function $F(z)$ can be represented in the following matrix form:

$$F(z) = \frac{1}{T'(z)} \sum_{k=0}^{N-1} R^{(k)} z^k,$$

where
$$(z-1)^{m(1)-1} \sum_{k=0}^{N-1} R_{i,j}^{(k)} z^k = M_{j,i},\ i,j = \overline{1,n}.$$

We will make some transformation using the identity $I = (I - zP)(I - zP)^{-1}$. We have
$$T'(z)I = (I - zP) \sum_{k=0}^{N-1} z^k R^{(k)} = \sum_{k=0}^{N-1} z^k R^{(k)} - \sum_{k=0}^{N-1} z^{k+1} (PR^{(k)}) =$$

$$= R^{(0)} + \sum_{k=1}^{N-1} z^k (R^{(k)} - PR^{(k-1)}) - z^N (PR^{(N-1)}).$$

Let $T'(z) = \sum\limits_{k=0}^{N} \beta_k^* z^k$ and substitute this expression in obtained above relation. Then we obtain the following formula for determining the matrices $R^{(k)},\ \ k = \overline{0, N-1}$:

$$R^{(0)} = \beta_0^* I; \quad R^{(k)} = \beta_k^* I + PR^{(k-1)},\ k = \overline{1, N-1}. \tag{10}$$

So, we have

$$F(z)=\left(\frac{V_{ij}(z)}{T'(z)}\right)_{i,j=\overline{1,n}},$$

where

$$V_{i,j}(z)=\sum_{k=0}^{N-1}R_{ij}^{(k)}z^k,\ \ i,j=\overline{1,n}.$$

Based on these formula we can develop algorithm for determining the matrix Q.

2.4 Polynomial Time Algorithm for Determining the Matrix of Limiting-State Probabilities Q

Consider

$$T(z)=\sum_{k=0}^{N-1}\gamma_k z^k;\ Y(z)=\sum_{k=0}^{N-2}y^{(k)}z^k;\ y^*=\alpha_1(1).$$

Then according to relation (8) we obtain

$$\frac{V_{i,j}(z)}{T'(z)}=F_{i,j}(z)=\frac{y^*_{i,j}}{z-1}+\frac{\sum\limits_{k=0}^{N-2}y_{ij}^{(k)}z^k}{T(z)},\ \ i,j=\overline{1,n}.$$

This involve

$$\sum_{k=0}^{N-1}R_{i,j}^{(k)}z^k=V_{i,j}(z)=y^*_{i,j}T(z)+(z-1)\sum_{k=0}^{N-2}y_{i,j}^{(k)}z^k=y^*_{i,j}\sum_{k=0}^{N-1}\gamma_k z^k+$$

$$+\sum_{k=0}^{N-2}y_{i,j}^{(k)}z^{k+1}-\sum_{k=0}^{N-2}y_{i,j}^{(k)}z^k=\sum_{k=0}^{N-1}\gamma_k y^*_{i,j}z^k+\sum_{k=1}^{N-1}y_{i,j}^{(k-1)}z^k-\sum_{k=0}^{N-2}y_{i,j}^{(k)}z^k=$$

$$=(\gamma_0 y^*_{i,j}-y_{i,j}^{(0)})+\sum_{k=1}^{N-2}(\gamma_k y^*_{i,j}+y_{i,j}^{(k-1)}-y_{i,j}^{(k)})z^k+(\gamma_{N-1}y^*_{i,j}+y_{i,j}^{(N-2)})z^{N-1},\ \ i,j=\overline{1,n}.$$

If we equate the corresponding coefficients of the variable z with the same exponents then we obtain the following system of linear equations:

$$\begin{cases} R_{i,j}^{(0)}=\gamma_0 y^*_{i,j}-y_{i,j}^{(0)}, \\ R_{i,j}^{(k)}=\gamma_k y^*_{i,j}+y_{i,j}^{(k-1)}-y_{i,j}^{(k)},\ \ k=\overline{1,N-2},\quad i,j=\overline{1,n}. \\ R_{i,j}^{(N-1)}=\gamma_{N-1}y^*_{i,j}+y_{i,j}^{(N-2)}, \end{cases}$$

This system is equivalent to the following system:

$$\begin{cases} y_{i,j}^{(0)} = \gamma_0 y_{i,j}^* - R_{i,j}^{(0)}, \\ \\ y_{ij}^{(k)} = \gamma_k y_{i,j}^* + y_{i,j}^{(k-1)} - R_{i,j}^{(k)},\ k = \overline{1, N-2},\quad i,j = \overline{1,n}. \\ \\ y_{i,j}^{(N-2)} = -\gamma_{N-1} y_{i,j}^* + R_{i,j}^{(N-1)}. \end{cases}$$

Here we can see that there exist the coefficients $u_{i,j}^{(k)},\ v_{i,j}^{(k)} \in \mathbb{C},\ k = \overline{0, N-2},\ \ i,j = \overline{1,n}$, such that

$$y_{i,j}^{(k)} = u_{i,j}^{(k)} y_{i,j}^* + v_{i,j}^{(k)},\ k = \overline{0, N-2},\ i,j = \overline{1,n}.$$

From the first equation we obtain

$$u_{i,j}^{(0)} = \gamma_0,\ v_{i,j}^{(0)} = -R_{i,j}^{(0)},\ i,j = \overline{1,n}.$$

From the next $N-2$ equations we obtain

$$y_{i,j}^{(k)} = \gamma_k y_{i,j}^* + y_{i,j}^{(k-1)} - R_{i,j}^{(k)} = \gamma_k y_{i,j}^* + u_{i,j}^{(k-1)} y_{i,j}^* + v_{i,j}^{(k-1)} - R_{i,j}^{(k)} =$$

$$= (\gamma_k + u_{i,j}^{(k-1)}) y_{i,j}^* + (v_{i,j}^{(k-1)} - R_{ij}^{(k)}),\ k = \overline{1, N-2},\ i,j = \overline{1,n},$$

which involve the recursive equations

$$u_{i,j}^{(k)} = u_{i,j}^{(k-1)} + \gamma_k,\ v_{ij}^{(k)} = v_{i,j}^{(k-1)} - R_{i,j}^{(k)},\ k = \overline{1, N-2},\ i,j = \overline{1,n}.$$

In a such way we obtain the direct formula for calculation of the coefficients:

$$u_{i,j}^{(k)} = \sum_{r=0}^{k} \gamma_r,\ v_{i,j}^{(k)} = -\sum_{r=0}^{k} R_{i,j}^{(r)},\ k = \overline{0, N-2},\ i,j = \overline{1,n}.$$

If we introduce these coefficients in the last equation of the system then we obtain

$$u_{i,j}^{(N-2)} y_{ij}^* + v_{i,j}^{(N-2)} = -\gamma_{N-1} y_{ij}^* + R_{i,j}^{(N-1)},\ i,j = \overline{1,n} \ \Leftrightarrow$$

$$\Leftrightarrow\ y_{i,j}^* \sum_{r=0}^{N-1} \gamma_r = \sum_{r=0}^{N-1} R_{i,j}^{(r)},\ i,j = \overline{1,n} \ \Leftrightarrow$$

$$\Leftrightarrow\ y_{i,j}^* = \frac{\sum\limits_{r=0}^{N-1} R_{i,j}^{(r)}}{\sum\limits_{r=0}^{N-1} \gamma_r} = \frac{R_{i,j}}{T(1)},\ i,j = \overline{1,n},$$

where $R_{ij} = \sum\limits_{r=0}^{N-1} R_{i,j}^{(r)},\ i,j = \overline{1,n}$. Finally, if we denote $R = (R_{ij})_{i,j=\overline{1,n}}$ then

$$Q = -\frac{1}{T(1)} R. \tag{11}$$

Based on result described above we can describe the algorithm for determining the matrix Q.

Algorithm 1.2: Determining the Limiting-State Matrix Q

1) Find the coefficients of the polynomial $\Delta(z) = \sum_{k=0}^{n} \beta_k z^k$ using Algorithm 1.1;

2) Divide $m(1)$ times the polynomial $\Delta(z)$ by $z-1$, using Horner scheme and find the polynomial $T(z)$ that satisfies the condition $T(1) \neq 0$. At the same time we preserve the coefficients β_k^*, $k = \overline{0,N}$, of the polynomial $T'(z) = (z-1)T(z)$ obtained at the previous step of the Horner's scheme;

3) Determine $T(1)$ according to the rule described above;

4) Find the matrices $R^{(k)}$, $k = \overline{0, N-1}$, according to (10);

5) Find the matrix $R = \sum_{k=0}^{N-1} R^{(k)}$;

6) Calculate the matrix Q according to formula (11);

It is easy to check that the running time of Algorithm 1.2 is $O(|X|^4)$. Indeed, step 1) and step 4) of the algorithm use $O(|X|^4)$ elementary operations and each of remainder steps 2) - 3) and 5) - 6) use in the worst case $O(|X|^3)$ elementary operations.

3 Numerical examples

In this section we give some numerical examples which illustrate the main details of the algorithms from previous section.

Example 1. Consider the discrete Markov process with the stochastic matrix of probability transactions $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We can see that $P^{n)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $P^{2n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\forall n \geq 0$, i.e. the Markov chain is 2-periodic.

So, in this case the limit $\lim\limits_{n\to\infty} P^n$ does not exist, but there exists the matrix Q which can be found by using algorithm described above. If we apply this algorithm then we obtain:

1) $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $s_1 = trP = 0$, $s_2 = trP^2 = 2$;

$$\beta_0 = 1,\ \beta_1 = -s_1 = 0,\ \beta_2 = -\frac{1}{2}(s_2 + \beta_1 s_1) = -1;$$

2) We divide the polynomial $\beta_2 z^2 + \beta_1 z + \beta_0$ by $z-1$ using Horner's scheme

	-1	0	1
1	-1	-1	0
1	-1	-2	

and obtain $m(1)=1,\ N=2;\ \beta_0^*=1,\ \beta_1^*=0,\ \beta_2^*=-1;\ \gamma_0=-1,\ \gamma_1=-1;$

3) $T(1)=\gamma_0+\gamma_1=-2;$

4) $R^{(0)}=\beta_0^* I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ R^{(1)}=\beta_1^* I+PR^{(0)}=\begin{pmatrix}0&1\\1&0\end{pmatrix};$

5) $R=R^{(0)}+R^{(1)}=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&1\\1&1\end{pmatrix};$

6) $Q=-\dfrac{1}{T(1)}R=\dfrac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}0.5&0.5\\0.5&0.5\end{pmatrix}.$

In such a way we obtain the limit matrix $Q=\begin{pmatrix}0.5&0.5\\0.5&0.5\end{pmatrix}$, however the considered process is not ergodic because the matrix $P^{(n)}$ contains zero elements $\forall n\geq 0$. The rows of this matrix are the same and the vector of limiting probabilities $\pi^*=(0.5,0.5)$ can be found also by solving the system of linear equation (4).

Example 2. Consider the Markov process with the stochastic matrix $P=\begin{pmatrix}0.5&0.5\\0.4&0.6\end{pmatrix}$. We can see that in this case the Markov process is ergodic. We can find the matrix Q using our algorithm:

$$P=\begin{pmatrix}0.5&0.5\\0.4&0.6\end{pmatrix},\ P^2=\begin{pmatrix}0.45&0.55\\0.44&0.56\end{pmatrix};$$

$$s_1=trP=0.5+0.6=1.1,\ s_2=trP^2=0.45+0.56=1.01;$$

$$\beta_0=1,\ \beta_1=-s_1=-1.1,\ \beta_2=-\frac{1}{2}(s_2+\beta_1 s_1)=-\frac{1}{2}(1.01-1.1\cdot 1.1)=0.1;$$

	0.1	-1.1	1
1	0.1	-1	0
1	0.1	-0.9	

$$\beta_0^*=1,\ \beta_1^*=-1.1,\ \beta_2^*=0.1;\ \gamma_0=-1,\ \gamma_1=0.1;\ T(1)=\gamma_0+\gamma_1=-0.9;$$

$$R^{(0)}=\beta_0^* I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ R^{(1)}=\beta_1^* I+PR^{(0)}=\begin{pmatrix}-0.6&0.5\\0.4&-0.5\end{pmatrix};$$

$$R=R^{(0)}+R^{(1)}=\begin{pmatrix}0.4&0.5\\0.4&0.5\end{pmatrix};\ Q=-\frac{1}{T(1)}R=\frac{1}{9}\begin{pmatrix}4&5\\4&5\end{pmatrix}.$$

We have $Q=\begin{pmatrix}4/9&5/9\\4/9&5/9\end{pmatrix}$. The rows of this matrix are the same and all elements of the matrix $P^{(n)}$ are non zero when $t\to\infty$. So, this is ergodoc Markov process with the vector of limiting probabilities $\pi_1^*=\dfrac{4}{9}$. As we have shown this vector can be found by solving system (4).

Example 3. We consider a non ergodic Markov process with the stochastic matrix of probabilities transactions

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

In this case the solution of the system of linear equations (4) is not unique. If we apply the proposed algorithm we can determine the matrix Q. According to this algorithm we obtain:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \ P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \end{pmatrix}, \ P^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{13}{27} & \frac{13}{27} & \frac{1}{27} \end{pmatrix};$$

$$s_1 = trP = 7/3, \ s_2 = trP^2 = 19/9, \ s_3 = trP^3 = 55/27; \ \beta_0 = 1,$$

$$\beta_1 = -s_1 = -7/3, \ \beta_2 = -(s_2 + \beta_1 s_1)/2 = 5/3, \ \beta_3 = -(s_3 + \beta_1 s_2 + \beta_2 s_1)/3 = -1/3;$$

	-1/3	5/3	-7/3	1
1	-1/3	4/3	-1	0
1	-1/3	1	0	
1	-1/3	2/3		

$$\beta_0^* = -1, \ \beta_1^* = 4/3, \ \beta_2^* = -1/3; \ \gamma_0 = 1, \ \gamma_1 = -1/3; \ T(1) = \gamma_0 + \gamma_1 = 2/3;$$

$$R^{(0)} = \beta_0^* I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ R^{(1)} = \beta_1^* I + PR^{(0)} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ -1/3 & -1/3 & 1 \end{pmatrix};$$

$$R = R^{(0)} + R^{(1)} = \begin{pmatrix} -2/3 & 0 & 0 \\ 0 & -2/3 & 0 \\ -1/3 & -1/3 & 0 \end{pmatrix}; \ Q = -\frac{1}{T(1)} R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

So, finally we have

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

In this case all rows of the matrix Q are different. It is easy to observe that for the considered example there exits $\lim\limits_{n\to\infty} P^{(n)} = Q$.

References

[1] Helmberg G., Voltkamp G. *On Fadeev-Leverrier's Method for the Computation of the Characteristic Polynomial of the Matrix and of Eigenvectors.* Linear Algebra and its Application, 1993, No. 185, 219–233.

[2] Howard R.A. *Dynamic Programming and Markov Processes.* Wiley, 1960.

[3] Lozovanu D., Pickl S. *Dynamic Programming Algorithms for Solving Stochastic Discrete Control Problems.* Bulletin of the Academy of Science of RM, Mathematics, 2009, No. 2(60), 73–90.

[4] Puterman M. *Markov Decision Processes.* Wiley, 1993.

[5] Paţiuc V., Râbacova G., Secrieru I., Zolotarevschi V. *Metode numerice în probleme diferenţiale, integrale şi de valori proprii.* Chişinău, 2001.

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