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The property of universality for some monoid algebras over non-commutative rings

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Abstract. We define on an arbitrary ring A a family of mappings $(\sigma_{x,y})$ subscripted with elements of a multiplicative monoid G. The assigned properties allow to call these mappings derivations of the ring A. A monoid algebra of G over A is constructed explicitly, and the universality property of it is shown.

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In this note we consider monoid algebras over non-commutative rings. First, we introduce axiomatically a family of mappings $\sigma = (\sigma_{x,y})$ defined on a ring A and subscripted with elements of a multiplicative monoid G. Due to their assigned properties these mappings can be called derivations of A. Next, we construct a monoid algebra $A\langle G\rangle$ by means of the family σ, and the universality of it is shown.

1. Let A be a ring (in general non-commutative) and G a multiplicative monoid. Throughout the paper we consider $1 \neq 0$ (where 0 is the null element of A, and 1 is the unit element for multiplication), the unit element of G is denoted by e. We introduce a family of mappings of A into itself by the following assumption.

(A) For each $x \in G$ there exists a unique family $\sigma_x = (\sigma_{x,y})_{y\in G}$ of mappings $\sigma_{x,y} : A \longrightarrow A$ such that $\sigma_{x,y} = 0$ for almost all $y \in G$ (here and thereafter, almost all will mean all but a finite number, that is, $\sigma_{x,y} \neq 0$ only for a finite set of $y \in G$) and for which the following properties are fulfilled:

(i) $\sigma_{x,y}(a + b) = \sigma_{x,y}(a) + \sigma_{x,y}(b)$ $(a, b \in A; x, y \in G)$;

(ii) $\sigma_{x,y}(ab) = \sum_{z\in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$ $(a, b \in A; x, y \in G)$;

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$ $(x, y, z \in G)$;

(iv_1) $\sigma_{x,y}(1) = 0$ $(x \neq y; x, y \in G)$; (iv_2) $\sigma_{x,x}(1) = 1$ $(x \in G)$;

(iv_3) $\sigma_{e,x}(a) = 0$ $(x \neq e; x \in G)$; (iv_4) $\sigma_{e,e}(a) = a$ $(a \in A)$.

In (ii) the elements are multiplied as in the ring A, but in (iii) the symbol $\circ$ means the composition of maps.

Examples. 1. Let A be a ring and let G be a multiplicative monoid, and let σ be a monoid-homomorphism of G into $End(A)$, i.e. $\sigma(xy) = \sigma(x) \circ \sigma(y)$ $(x, y \in G)$ and $\sigma(e) = 1_A$. We define $\sigma_{x,y} : A \longrightarrow A$ such that $\sigma_{x,x} = \sigma(x)$ for $x \in G$ and $\sigma_{x,y} = 0$ for $y \neq x$. The properties $(i) - (iv_4)$ of (A) are verified at once.

2. Let A be a ring, and let α be an endomorphism of A and δ be an α-differentiation of A, i.e.

$$\delta(a+b) = \delta(a) + \delta(b),\ \delta(ab) = \delta(a)b + \alpha(a)\delta(b)$$

for every $a, b \in A$. Denote by G the monoid of elements x_n $(n = 0, 1, ...)$ endowed with the law of composition defined by $x_n x_m = x_{n+m}$ $(n, m = 0, 1, ...;\ x_0 := e)$. We write σ_{nm} instead of σ_{x_n, x_m} by defining $\sigma_{nm} : A \longrightarrow A$ as the following mappings $\sigma_{00} = 1_A,\ \sigma_{10} = \delta,\ \sigma_{11} = \alpha,\ \sigma_{nm} = 0$ for $m > n$ and $\sigma_{nm} = \sum_{j_1+...+j_n=m} \sigma_{1j_1} \circ ... \circ \sigma_{1j_n}$ $(m = 0, 1, ..., n;\ n = 1, 2, ...)$, where $j_k = 0, 1$ $(k = 1, ..., n)$. The family $\sigma = (\sigma_{nm})$ satisfies the axioms $(i) - (iv_4)$ of (A).

2. Next, we consider an algebra $A\langle G\rangle$ connected with the structure of differentiation $\sigma = (\sigma_{x,y})$. Let $A\langle G\rangle$ be the set of all mappings $\alpha : G \longrightarrow A$ such that $\alpha(x) = 0$ for almost all $x \in G$. We define the addition in $A\langle G\rangle$ to be the ordinary addition of mappings into the additive group of A and define the operation of A on $A\langle G\rangle$ by the map $(a, \alpha) \longrightarrow a\alpha$ $(a \in A)$, where $(a\alpha)(x) = a\alpha(x)$ $(x \in G)$. Note that, in respect to these operations, $A\langle G\rangle$ forms a left module over A. Following notations made in [1] we write an element $\alpha \in A\langle G\rangle$ as a sum $\alpha = \sum_{x\in G} a_x \cdot x$, where by $a \cdot x$ $(a \in A,\ x \in G)$ is denoted the mapping whose value at x is a and 0 at elements different from x. Certainly, the above sum is taken over almost all $x \in G$. $A\langle G\rangle$ becomes a ring if for elements of the form $a \cdot x$ $(a \in A; x \in G)$ we define their product by the rule

$$(a \cdot x)(b \cdot y) = \sum_{z\in G} a\sigma_{x,z}(b) \cdot zy\ (a, b \in A; x, y \in G)$$

and then extend for $\alpha, \beta \in A\langle G\rangle$ by the property of distributivity. We let

$$\alpha a = \sum_{x\in G} \Big(\sum_{y\in G} a_y \sigma_{y,x}(a) \Big) \cdot x, \quad (a \in A, alpha \in A\langle G\rangle)$$

for $a \in A$ and $\alpha \in A\langle G\rangle$, and thus we obtain an operation of A on $A\langle G\rangle$ and in such a way we make $A\langle G\rangle$ into a right A-module. Thus, we may view $A\langle G\rangle$ as an algebra over A.

Remark. Let us consider the situation described in Example 1. Then the law of multiplication in $A\langle G\rangle$ is given as follows

$$\Big(\sum_{x\in G} a_x \cdot x \Big) \Big(\sum_{x\in G} b_x \cdot x \Big) = \sum_{x\in G} \sum_{y\in G} a_x \sigma_{x,x}(b_y) \cdot xy.$$

In this case, the monoid algebra $A\langle G\rangle$ represents a crossed product [2, 3] of the multiplicative monoid G over the ring A with respect to the factors $\rho_{x,y} = 1$ $(x, y \in G)$. If G is a group, and $\sigma : G \longrightarrow End(A)$ is such that $\sigma(x) = 1_A$ for all $x \in G$, we evidently obtain an ordinary group ring [4] (the commutative case see also [5]).

3. In this subsection we show that $A\langle G\rangle$ is a free G - algebra over A. Let B be another ring. Given a ring-homomorphism $f : A \longrightarrow B$ it can be defined on the ring B a structure of A-module, defining the operation of A on B by the map $(a,b) \longrightarrow f(a)b$ for all $a \in A$ and $b \in B$. We denote this operation by $a * b$. The axioms for a module are trivially verified. Let now $\varphi : G \longrightarrow B$ be a multiplicative monoid-homomorphism. Denote by $\langle B; f, \varphi\rangle$ the module formed by all linear combinations of elements $\varphi(x)$ $(x \in G)$ over A in respect to the operation $*$. The axioms for a left A-module are trivially verified.

We assume that the homomorphisms f and φ satisfy the following assumption.

(B) $\varphi(G)f(A) \subset \langle B; f, \varphi\rangle$.

Thus, it is postulated that an element $\varphi(x)f(a)$ $(a \in A, x \in G)$ can be written as a linear combination of the form $\sum_{b\in B, y\in G} b\,\varphi(y)$. The coefficients b depend on $\varphi(x), \varphi(y)$ and $f(a)$. To designate this fact we denote the corresponding coefficients by $\sigma_{\varphi(x),\varphi(y)}(f(a))$. Therefore, it can be considered that there are defined a family of mappings $\sigma_{\varphi(x),\varphi(y)} : B \longrightarrow B$ such that

$$\varphi(x)f(a) = \sum_{y\in G} \sigma_{\varphi(x),\varphi(y)}(f(a))\varphi(y) \ (a \in A, x \in G).$$

By these considerations, we may view $\langle B; f, \varphi\rangle$ as a right A-module. In order to make the module $\langle B; f, \varphi\rangle$ to be a ring we require the following additional assumption.

(C) The homomorphisms f and φ are such that the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ A \arrow[u, "\sigma_{x,y}"] \arrow[r, "f"] & B \arrow[u, "\sigma_{\varphi(x),\varphi(y)}"'] \end{tikzcd}$$

is commutative for every $x, y \in G$, i.e. $\sigma_{\varphi(x),\varphi(y)} \circ f = f \circ \sigma_{x,y}$ $(x, y \in G)$.

We define multiplication in $\langle B; f, \varphi\rangle$ by the rules

$$\Big(\sum_{x\in G} a_x * \varphi(x)\Big)\Big(\sum_{x\in G} b_x * \varphi(x)\Big) = \sum_{x\in G}\sum_{y\in G} (a_x * \varphi(x))(b_y * \varphi(y)),$$

$$(a_x * \varphi(x))(b_y * \varphi(y)) = f(a_x) \sum_{z\in G} \sigma_{\varphi(x),\varphi(z)}(f(b_y))\varphi(zy).$$

The verification that $\langle B; f, \varphi\rangle$ is a ring under the above laws of composition is direct. Thus, we have made $\langle B; f, \varphi\rangle$ into an algebra over A (in general, non-commutative).

Next, we define a category $\mathcal{C}$ whose objects are algebras $\langle B; f, \varphi\rangle$ constructed as above, and whose morphisms between two objects $\langle B; f, \varphi\rangle$ and $\langle B'; f', \varphi'\rangle$ are ring-homomorphisms $h : B \longrightarrow B'$ making the diagrams commutative:

$$\begin{tikzcd} G \arrow[r, equal] \arrow[d, "\varphi"'] & G \arrow[d, "\varphi'"] \\ B \arrow[r, "h"] & B' \\ A \arrow[u, "f"] \arrow[r, equal] & A \arrow[u, "f'"'] \end{tikzcd}$$

The axioms for a category are trivially satisfied. We call a universal object in the category $\mathcal{C}$ a free G-algebra over A, or a free (A, G)-algebra. It turns out that the monoid algebra $A\langle G\rangle$ represents a free (A, G)-algebra. To this end, we observe that the mapping $\varphi_0 : G \longrightarrow A\langle G\rangle$ given by $\varphi_0(x) = 1 \cdot x$ $(x \in G)$ is a monoid-homomorphism. The mapping φ_0 is embedding of G into $A\langle G\rangle$. In addition, we have a ring-homomorphism $f_0 : A \longrightarrow A\langle G\rangle$ given by $f_0(a) = a \cdot e$ $(a \in A)$. Obviously, f_0 is also an embedding. We identify $A\langle G\rangle$ with the triple $\langle A\langle G\rangle; f_0, \varphi_0\rangle$ and in this sense we treat $A\langle G\rangle$ as an object of the category $\mathcal{C}$. The property of the universality of $A\langle G\rangle$ is formulated by the following assertion.

Theorem 1. *Let A be a ring, and G a multiplicative monoid for which the assumptions $(A), (B)$ and (C) are satisfied. Then for every object $\langle B; f, \varphi\rangle$ of the category $\mathcal{C}$ there exists a unique ring-homomorphism $h : A\langle G\rangle \longrightarrow B$ making the following diagram commutative*

$$\begin{array}{ccc}
G & === & G \\
{\scriptstyle\varphi_0}\downarrow & & \downarrow{\scriptstyle\varphi} \\
A\langle G\rangle & \xrightarrow{h} & B \\
{\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle f} \\
A & === & A
\end{array}$$

The relation with the theory of skew polynomial rings [6–8] and with those obtained by Yu. M. Ryabukhin [9] (see also [10]), and further properties of the general derivation mappings $\sigma_{x,y}$ $(x, y \in G)$ will be given in a subsequent publication.

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