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About commutative Moufang loops of finite special rank

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Abstract. Commutative Moufang loops of finite special rank are characterized with the help of various associative subloops and with the help of various abelian subgroups of their multiplication group.

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By analogy with group theory [1], the (*special*) *rank* of loop Q is the least positive number rQ with the following property: any finitely generated subloop of loop Q can be generated by rQ elements; if there are not such numbers, then we sat $rQ = \infty$. As to the structure and subject is concerned this paper is analogous to paper [2], where the commutative Moufang loops (abbreviated CMLs) with maximum condition for subloops are characterized. All notions and results of the CML theory we need are in detail described in the paper [2] (see also [3]). The present work characterizes a CML of finite rank with the help of various associative subloops and with the help of various abelian subgroups of its multiplication group.

Takes place

Lemma 1. *The following statements are equivalent for an arbitrary CML Q:*

1) CML Q satisfies the minimum condition for subloops;

2) CML Q is a direct product of a finite number of quasicyclic groups, belonging to the center of CML Q, and a finite CML;

3) CML Q satisfies the minimum condition for invariant subloops;

4) CML Q satisfies the minimum condition for non-invariant associative subloops;

5) if the CML Q contains a centrally nilpotent of class n subloop, then it satisfies the minimum condition for centrally nilpotent of class n subloops;

6) if the CML Q contains a centrally solvable of class s subloop, then it satisfies the minimum condition for centrally solvable of class s subloops;

7) at least one maximal associative subloop of the CML Q satisfies the minimum condition for subloops.

The equivalence of conditions 1), 2), 3) is proved in [4], the equivalence of conditions 1), 4), 5), 6) is proved in [5] and the equivalence of conditions 1), 7) is proved in [6].

Lemma 2. *If H is an invariant subloop of an arbitrary loop Q, then*

$$rQ \leq rH + r(Q/H).$$

Proof. Let t, s be ranks of loops H and Q/H respectively. Let us take in Q a finite set M and consider the subloop $xH|x \in M >$ in the quotient loop Q/H. As it is finitely generated, then it is possible to select in loop $< M >$ such a subset S of order s that

$$< xH|x \in M >=< xH|x \in S > .$$

Let us fix some notation for any $x \in M$

$$x = x_s x_h, x_s \in < S >, x_h \in < H > .$$

As $< x_h|x \in M >$ is a finitely generated subloop of loop H, then it is generated by a certain subset T of order $\leq t$. Obviously, $< M >=< T \cup S >$, i.e. $rQ \leq t + s$. This completes the proof of Lemma 2.

Further we will need a well known statement.

Lemma 3. *A primary abelian group has a finite rank r if and only if it decomposes into a direct product of r cyclic and quasicyclic groups and, consequently, satisfies the minimum condition for subgroups.*

We will also need the marvellous theorem of M.I. Kargapolov [7].

Lemma 4. *Let A be an invariant abelian subgroup of group G. If all abelian subgroups of group G have finite ranks, then the abelian subgroups of group G/A also have finite ranks.*

Lemma 5 *A periodic CML has a finite rank if and only if it satisfies the maximum condition for subloops.*

Proof. An arbitrary periodic CML Q decomposes into a direct product of its maximal p-subloops Q_p. In addition, for $p \neq 3$, Q_p are abelian groups [8]. Therefore, if Q has a finite rank rQ, then the number of subloops Q_p is finite and ranks rQ_p do not exceed the rank rQ. Conversely, if the CML Q satisfies the minimum condition for subloops, then the number of subloops Q_p is finite and each subloop Q_p satisfies the minimum condition for subloops. Hence it is enough to consider the case when the CML Q is a 3-loop.

Let us suppose that the CML Q has a finite rank. Then all its associative subloops have finite ranks and, by Lemma 3, satisfy the minimum condition for subloops. Then by 5) of Lemma 1 the CML Q satisfies the minimum condition for subloops as well.

Conversely, if the CML Q satisfies the minimum condition for subloops, then by 2) of Lemma 1 it decomposes into a direct product of a finite number of quasicyclic groups and a finite CML. Then it follows from Lemma 3 that the CML Q has a finite rank. This completes the proof of Lemma 5.

Lemma 6 *Let Q be a non-periodic CML with the center $Z(Q)$ of finite rank. Then $Z(Q)$ decomposes into the direct product $Z(Q) = T \times L$, where T is a non-periodic abelian group, L is an abelian group without torsion, and the quotient loop Q/L is a periodic CML.*

Proof. It is shown in [3] that for any element $x \in Q$, $x^3 \in Z(Q)$. Hence the center $Z(Q)$ is a non-periodic abelian group. The periodic part T of group $Z(Q)$ decomposes into a direct product of its primary components and as T has a finite rank, their number is finite. Therefore T has a finite exponent. Then by Corollary 10.1.13 from [9] T stands as a direct factor the $Z(Q) = T \times L$, where L is an abelian group without torsion. CML $Q/Z(Q)$ has the exponent three. Then it follows from the relations $Q/Z(Q) \cong (Q/L)/(Z(Q)/L) = (Q/L)/((T \times L)/L) \cong (Q/L)/T$ that the CML Q/L is periodic as the extension of the periodic CML T with the help of the periodic CML $Q/Z(Q)$. This completes the proof of Lemma 6.

We remind that the *multiplication group* $\mathfrak{M}(Q)$ of an arbitrary CML Q is the group, generated by all permutations $L(x)$, where $L(x)y = xy$.

Lemma 7. *A CML Q has a finite rank rQ if and only if the rank $r\mathfrak{M}$ of its multiplication group $\mathfrak{M}$ is finite.*

Proof. Let a CML Q have a finite rank and let $\mathfrak{N}$ be a subgroup of the group $\mathfrak{M}$, generated by a finite set of elements $\varphi_1, \dots, \varphi_t$. By the definition of group $\mathfrak{M}$, any element φ_i $(i = 1, \dots, t)$ is a product of a finite number of permutations $L(a_{il})$. Let H be a subloop of CML Q, generated by all elements a_{ij}. The subloop H can be generated by no more that rQ elements $b_1, \dots, b_m$. Let us now suppose that element a_{ij}, written via the generators $b_1, \dots, b_m$, has the form $a_{ij} = uv$, where u, v contain in the notation less generators $b_1, \dots, b_m$ than a_{ij}. Let us show that the subgroup $\mathfrak{N}$ is generated by the permutations $L(b_1), \dots, L(b_m)$. Indeed, the CLM is characterized by the identity $x^2 \cdot yz = xy \cdot yz$ [3]. Then $L(x^2)L(y)z = L(xy)L(x)z$, $L(x^2)L(y) = L(xy)L(x)$, $L(xy) = L(x^2)L(y)L^{-1}(x)$, $L(xy) = L(x)L(x)L(y)L^{-1}(x)$. Hence $L(a_{ij}) = L(uv) = L(u)L(u)L(v)L^{-1}(u)$. Continuing this process with the permutations $L(u), L(v)$, we will obtain after a finite number of steps that the permutation $L(a_{ij})$ is expressed through the permutations $L(b_1), \dots, L(b_m)$. Consequently, $r\mathfrak{M} \leq rQ$.

Let now H be a finitely generated subloop of CML Q, let $\{h_1, \dots, h_k\}$ be a certain minimal system of its generators and let the rank $r\mathfrak{M}$ be finite. Let us now consider the subgroup $\mathfrak{N}$ of group $\mathfrak{M}$ with a system of generating elements $\{L(h_1), \dots, L(h_k)\}$. This system is minimal because if any of the generators is expressed through the others, for example $L(h_1) = L^{-1}(h_2)L(h_3)$, then $L(h_1)1 = L^{-1}(h_2)L(h_3)1$, $h_1 = h_2^{-1}h_3$, thus contradicting the minimality of system $\{h_1, \dots, h_k\}$. As the group $\mathfrak{M}$ is locally nilpotent [4], then the commutator-group $\mathfrak{M}'$ of group $\mathfrak{M}$ is contained into the Frattini subgroup. Then the quotient group $\mathfrak{M}/\mathfrak{M}'$ decomposes into a direct product of cyclic groups and their number coincides with the number of the generators of a minimal system [9]. Therefore any minimal system of generators has the same number of elements. Consequently, the system $\{h_1, \dots, h_k\}$ contains no more than $r\mathfrak{M}$ elements, hence $rQ \leq r\mathfrak{M}$. This completes the proof of Lemma 7.

Lemma 8. *The multiplication group $\mathfrak{M}$ of a periodic CML Q has a finite rank if and only if it satisfies the minimum condition for subloops.*

Proof. The multiplication group $\mathfrak{M}$ of a periodic CML Q is locally nilpotent and decomposes into a direct product of its maximal p-subgroups $\mathfrak{M}_p$ [4]. If the group $\mathfrak{M}$ has a finite rank or satisfies the minimum condition for subgroups, then it is obvious that the number of subgroups $\mathfrak{M}_p$ is finite. Let the group $\mathfrak{M}$ have a finite rank. Then each subgroup $\mathfrak{M}_p$ has a finite rank. It is shown in [10] that a locally nilpotent p-group has a finite rank if and only if it satisfies the minimum condition for subgroups. Therefore subgroups $\mathfrak{M}_p$, and then the group $\mathfrak{M}$ too, satisfy the minimum condition for subgroups. Conversely, if the group $\mathfrak{M}$ satisfies the minimum condition for subgroups, then each subgroup $\mathfrak{M}_p$ satisfies this condition. Hence they have finite ranks, then the group $\mathfrak{M}$ has a finite rank as well.

Lemma 9. *Let $\mathfrak{A}$ be the maximal abelian subgroup of the multiplication group $\mathfrak{M}$ of a CML. Then any non-unitary invariant subgroup $\mathfrak{N}$ of the group $\mathfrak{M}$ has a non-unitary intersection with $\mathfrak{A}$.*

Proof. Let us suppose the contrary, that $\mathfrak{A} \cap \mathfrak{N} = \{e\}$. As $\mathfrak{A}$ is the maximal abelian subgroup, then there exist such elements $\alpha \in \mathfrak{A}, \nu \in \mathfrak{N}$, that $[\alpha, \nu] \neq e$. The subgroup $\mathfrak{N}$ is invariant in $\mathfrak{M}$, therefore $[\alpha, \nu] \in \mathfrak{N}$. The commutator-group $\mathfrak{M}'$ is a 3-group [3], then there exists such an element $\beta \in \mathfrak{N}$ that $\beta^3 = e$. We denote $\mathfrak{C} =< \mathfrak{A}, \beta >$. The group $\mathfrak{M}$ is locally nilpotent [4] and as the subgroup $\mathfrak{A}$ is maximal in $\mathfrak{C}$, then it is invariant in $\mathfrak{C}$ [9]. The subgroup $\mathfrak{N}$ is invariant in $\mathfrak{M}$, then the subgroup $\mathfrak{C} \cap \mathfrak{N}$ is invariant in $\mathfrak{C}$ as well. By the supposition $\mathfrak{A} \cap \mathfrak{N} = \{e\}$, there are two different the unitary element invariant in $\mathfrak{C}$ subgroups $\mathfrak{C} \cap \mathfrak{N}$ and $\mathfrak{A}$ intersect on the unitary element. Therefore $\mathfrak{C}$ is a direct product of the groups $\mathfrak{C} \cap \mathfrak{N}$ and $\mathfrak{A}$. Then any element of $\mathfrak{A}$ commutes with elements of $\mathfrak{C} \cap \mathfrak{N}$. But it contradicts the maximality of the abelian group $\mathfrak{A}$. This completes the proof of Lemma 9.

Proposition 1. *The following conditions are equivalent for an arbitrary non-associative CML Q with the multiplication group $\mathfrak{M}$:*

1) the group $\mathfrak{M}$ satisfies the minimum condition for subgroups;

2) the group $\mathfrak{M}$ satisfies the minimum condition for non-invariant subgroups;

3) at least one maximal abelian subgroup of the group $\mathfrak{M}$ satisfies the minimum condition for subgroups.

Proof. The quotient group $\mathfrak{M}/Z(\mathfrak{M})$ is a 3-group [3]. Hence, if α is a certain element of infinite order from $\mathfrak{M}$, then for a certain n, $\alpha^n \in Z(\mathfrak{M})$. If the group $\mathfrak{M}$ satisfies the condition 2) or 3), then there is an abelian subgroup $\mathfrak{A}$ in $\mathfrak{M}$ which contains the center $Z(\mathfrak{M})$ and satisfies the minimum condition for subgroups. Therefore the group $\mathfrak{M}$ is periodic. Then $\mathfrak{M}$ decomposes into a direct product of its maximal p-subgroups $\mathfrak{M}_p$, in addition for $p \neq 3$, $\mathfrak{M}_p \subseteq Z(\mathfrak{M})$. It follows from the definition of group $\mathfrak{A}$ that $\mathfrak{M}_p \subseteq \mathfrak{A}$ for $p \neq 3$. Therefore it is enough to consider the case when $\mathfrak{M}$ is a 3-group.

Let now the condition 2) hold in the group $\mathfrak{M}$ and let us suppose that the condition 1) does not hold in $\mathfrak{M}$. Then it follows from Theorem 4.11 from [11] that the group $\mathfrak{M}$ contains an invariant subgroup $\mathfrak{A}$, within which all its cyclic subgroups are invariant in $\mathfrak{M}$, such that the quotient group $\mathfrak{M}/\mathfrak{A}$ is a cyclic group. Let us suppose that the CML Q is generated by the elements $b, a_1, a_2, \ldots$. Taking into account the construction of group $\mathfrak{M}$ we will suppose that the permutations $L(a_i)$, $i = 1, 2, \ldots$ (perhaps $L(b)$ too) belong to subgroup $\mathfrak{A}$. Hence they generate in $\mathfrak{M}$ invariant subgroups. Then for an arbitrary fixed element $x \in Q$ and a certain natural number n, $L^{-1}(x)L(a_i)L(x) = L^n(a_i)$. Further, $L(a_i)L(x) = L(x)L^n(a_i)$, $L(a_i)L(x)y = L(x)L^n(a_i)y$, $a_i \cdot xy = a \cdot a_i^n y$. If $y = 1$, then $a_i x = x a_i^n$, $x a_i = x a_i^n, a_i^{n-1} = 1$. The last equality holds true only for $n = 1$. Then $a_i \cdot xy = x \cdot a_i y$, i.e. $a_i \in Z(Q)$. It follows easily from here that the CML Q is associative. Contradiction. Consequently, the CML Q satisfies the condition 1).

Let us now suppose that the condition 3) holds true in group $\mathfrak{M}$. The group $\mathfrak{M}$ is locally nilpotent [4], then by Theorem 1.8 from [11] it has a central system. Now, taking into account Lemma 9, the further proof of implication 3) $\rightarrow$ 1) can be completed exactly repeating the proof of Theorem 1.19 from [11]. Further, as the implications 1) $\rightarrow$ 2), 1) $\rightarrow$ 3) are obvious, the proposition is proved.

Proposition 2. *The following conditions are equivalent for an arbitrary non-associative commutative Moufang ZA-loop Q:*

1) CML Q has a finite rank;

2) if the CLM Q contains an invariant (resp. non-invariant) centrally nilpotent of class n subloop, then all invariant (or non-invariant) associative subloops of invariant (resp. non-invariant) centrally nilpotent of class n subloops of the CML Q have finite ranks;

3) if the CLM Q contains an invariant (resp. non-invariant) centrally solvable of class s subloop, then all invariant (or non-invariant) associative subloops of invariant (resp. non-invariant) centrally solvable of class s subloops of the CML Q have finite ranks;

4) if the CLM Q contains an invariant (resp. non-invariant) centrally nilpotent of class n subloop, then all invariant (or non-invariant) associative subloops of at least one maximal invariant (resp. non-invariant) centrally nilpotent of class n subloops of the CML Q have finite ranks;

5) if the CLM Q contains an invariant (resp. non-invariant) centrally solvable of class s subloop, then all invariant (or non-invariant) associative subloops of at least one maximal invariant (resp. non-invariant) centrally solvable of class s subloops of the CML Q have finite ranks;

6) the center $Z(Q)$ of CML Q has a finite rank;

7) the group $\mathfrak{M}$ has a finite rank;

8) if the group $\mathfrak{M}$ contains an invariant (resp. non-invariant) nilpotent of class n subgroup, then all invariant (or non-invariant) abelian subgroups of invariant (resp. non-invariant) nilpotent of class n subgroups of group $\mathfrak{M}$ have finite ranks;

9) if the group $\mathfrak{M}$ contains an invariant (resp. non-invariant) solvable of class s subgroup, then all invariant (or non-invariant) abelian subgroups of invariant (resp.

non-invariant) solvable of class s subgroup of group $\mathfrak{M}$ have finite ranks;

10) if the group $\mathfrak{M}$ contains an invariant (resp. non-invariant) nilpotent of class n subgroup, then all invariant (or non-invariant) abelian subgroups of at least one maximal invariant (resp. non-invariant) nilpotent of class n subgroups of group $\mathfrak{M}$ have finite ranks;

11) if the group $\mathfrak{M}$ contains an invariant (resp. non-invariant) solvable of class s subgroup, then all invariant (or non-invariant) abelian subgroups of at least one maximal invariant (resp. non-invariant) solvable of class s subgroup of group $\mathfrak{M}$ have finite ranks;

12) the center $Z(\mathfrak{M})$ of group $\mathfrak{M}$ has a finite rank.

Proof. It is easy to notice that if the CML Q satisfies one of the conditions 2) – 5), then there are invariant (or non-invariant) associative subloops of finite rank in Q which contain the center $Z(Q)$. Therefore the implications 1) $\rightarrow$ 2) $\rightarrow$ 6), 1) $\rightarrow$ 3) $\rightarrow$ 6), 1) $\rightarrow$ 4) $\rightarrow$ 6), 1) $\rightarrow$ 5) $\rightarrow$ 6) hold true. By analogy the implications 7) $\rightarrow$ 8) $\rightarrow$ 12), 7) $\rightarrow$ 9) $\rightarrow$ 12), 7) $\rightarrow$ 10) $\rightarrow$ 12), 7) $\rightarrow$ 11) $\rightarrow$ 12) also hold true. Further, it is shown in [2] that the relation $Z(Q) \cong Z(\mathfrak{M})$ holds true for an arbitrary CML Q. Hence the implications 6) $\leftrightarrow$ 12) hold as well.

Let us now prove the justice of implication 6) $\rightarrow$ 1). Let us suppose that CML Q is periodic. The center $Z(Q)$ has a finite rank, then by Lemma 5 it satisfies the minimum condition for subloops. It is shown in [6] that the ZA-loop Q also satisfies this condition, and the justice of condition follows from Lemma 1.

However, if CML Q is non-periodic, then it follows from Lemma 6 that $Z(Q) = T \times L$, where T is a periodic abelian group, L is an abelian group without torsion, and that Q/L is periodic. It is shown in [2] that under the homomorphism $Q \rightarrow Q/L$ the center $Z(Q)$ is mapped in the center $Z(Q/L)$. But $Z(Q)/L = (T \times L)/L \cong T$. Hence the center $Z(Q/L)$ of the periodic CML Q/L has a finite rank. Then by the previous case CML Q/L has a finite rank too. As subloop L has a finite rank, by Lemma 2 the CML Q has a finite rank too, i.e. the condition 1) holds true in Q.

Finally, the implication 1) $\rightarrow$ 7) follows from Lemma 7.

Theorem 1. *The following conditions are equivalent for an arbitrary non-associative CML Q with the multiplication group $\mathfrak{M}$:*

1) CML Q has a finite rank;

2) if the CLM Q contains a centrally nilpotent of class n subloop, then all invariant (or non-invariant) associative subloops of centrally nilpotent of class n subloops of CML Q have a finite rank;

3) if the CLM Q contains a centrally solvable of class s subloop, then all non-invariant associative subloops of centrally solvable of class s subloops of CML Q have a finite rank;

4) non-invariant associative subloops of invariant subloops of CML Q has a finite rank;

5) at least one maximal associative subloop of CML Q has a finite rank;

6) the group $\mathfrak{M}$ has a finite rank;

7) if the group $\mathfrak{M}$ contains a nilpotent of class n subgroup, then all non-invariant abelian subgroups of nilpotent of class n subgroups of group $\mathfrak{M}$ have a finite rank;

8) if the group $\mathfrak{M}$ contains a solvable of class s subgroup, then all non-invariant abelian subgroups of solvable of class s subgroups of group $\mathfrak{M}$ have a finite rank;

9) non-invariant abelian subgroups of invariant subgroups of group $\mathfrak{M}$ has a finite rank;

10) at least one maximal abelian subgroup of the group $\mathfrak{M}$ has a finite rank.

Proof. Implications 1) $\rightarrow$ 2), 1) $\rightarrow$ 3), 1) $\rightarrow$ 4), 1) $\rightarrow$ 5) are obvious. Let us prove the justice of the converse implications. Let us firstly consider the case when the CML Q is periodic. Any centrally nilpotent CML is a ZA-loop. Hence it follows from 2) of Proposition 2 that if CML Q satisies the condition 2), then all its centrally nilpotent of class n subloops have finite ranks. By Lemma 5 they satisfy the minimum condition for subloops. Then by Lemma 1 the CML Q satisfies this condition too, and it follows from Lemma 5 that the condition 1) holds in CML Q.

If the condition 3) (resp. 4)) holds in CML Q, then it follows from Lemma 5 and 4) of Lemma 1 that centrally solvable of class s (resp. invariant) subloops of CML Q satisfy the minimum condition for subloops in CML Q. Then by Lemma 1 the CML Q satisfies this condition as well, and it follows from Lemma 5 that the condition 1) holds in CML Q. Implication 5) $\rightarrow$ 1) is proved by analogy.

Let now CML Q be non-periodic. It is easy to notice that the statements hold true. If the CML Q satisfies one of the conditions 2) – 5), then in CML Q there are associative subloops of finite rank, containing $Z(Q)$. Therefore the center $Z(Q)$ has a finite rank. By analogy, if CML Q satisfies one of the conditions 7) – 10), then the center $Z(\mathfrak{M})$ of group $\mathfrak{M}$ has a finite rank.

Let us now suppose that one of the conditions 2) – 5) holds true in CML Q. Let L be an abelian group without torsion, considered in Lemma 6. It is proved in [2] that if a subloop H of CML Q is centrally nilpotent of class n or centrally solvable of class s, or maximally associative, or invariant, either non-invariant, then the image of subloop H under the homomorphism $Q \rightarrow Q/L$ will be of the same type. The abelian group L has a finite rank. If K is an associative subloop of finite rank of CML Q, containing L, then it follows from Lemma 3 that the abelian group K/L has also a finite rank. It follows from here that if the CML Q satisfies one of the conditions 2) – 5), then CML Q/L will satisfy this condition as well. By Lemma 6 the CML Q/L is periodic. Then by the previous case the CML K/L has a finite rank. Further, as the subloop L has a finite rank, then it follows from Lemma 2 that the CML Q has a finite rank as well. Consequently, the implications 2) $\rightarrow$ 1), 3) $\rightarrow$ 1), 4) $\rightarrow$ 1), 5) $\rightarrow$ 1) hold true.

Implications 1) $\leftrightarrow$ 6) are Lemma 7. Implications 6) $\rightarrow$ 7), 6) $\rightarrow$ 8), 6) $\rightarrow$ 9), 6) $\rightarrow$ 10) are obvious. Let us prove the converse implications. It follows from Proposition 1 that conditions 7), 8), 9) hold true not only for non-invariant abelian subgroups, but for all abelian subgroups as well. It is shown in [12] that if in a locally nilpotent group all abelian subgroups have finite ranks, then the group itself has a finite rank. The multiplication group $\mathfrak{M}$ is locally nilpotent [4]. Therefore if the condition 7) (resp. 8) or 9)) holds true in CML Q, then all nilpotent of class n subgroups (resp. all solvable of class s subgroups, or all invariant subgroups, either at least one maximal abelian subgroup) of group $\mathfrak{M}$ have finite ranks. Let us suppose that the CML Q is periodic. Then by Lemma 8 all nilpotent of class n subgroups (resp. all solvable of class s subgroups, or all invariant subgroups, either at least one maximal abelian subgroup) of group $\mathfrak{M}$ satisfy the minimum condition for subgroups. It is shown in [13, 14] that a locally nilpotent group, containing a nilpotent of class n subgroup (or solvable of class s subgroup) and satisfying the minimum condition for nilpotent of class n subgroups (or solvable of class s subgroups), satisfies the minimum condition for subgroups. If a locally nilpotent group satisfies the minimum condition for invariant subgroups, then it satisfies the minimum condition for subgroups too [15]. Therefore, if we take into account Proposition 1, then the multiplication group $\mathfrak{M}$ satisfies the minimum condition for subgroups and by Lemma 8, it has a finite rank. Consequently, if the CML Q is periodic, then the implications 7) $\rightarrow$ 6), 8) $\rightarrow$ 6), 9) $\rightarrow$ 6), 10) $\rightarrow$ 6) hold true.

It is obvious that if the CML Q satisfies one of the conditions 7) – 10), in the group $\mathfrak{M}$ there are abelian groups of finite rank, containing the center $Z(\mathfrak{M})$. Hence $Z(\mathfrak{M})$ has a finite rank. It is shown in [2] that $Z(\mathfrak{M}) \cong Z(Q)$. Then it follows from Lemma 6 that the group $Z(\mathfrak{M})$ decomposes into a direct product of periodic abelian group $\mathfrak{T}$ and abelian group without torsion $\mathfrak{L}$. The CML Q is non-periodic. Then the group $\mathfrak{M}$ is also non-periodic [4]. It is shown in [3] that the quotient group $\mathfrak{M}/Z(\mathfrak{M})$ is a 3-group. Hence, if α is an element of infinite order from $\mathfrak{M}$, then $\alpha^n \in \mathfrak{L}$. Therefore the group $\mathfrak{M}/\mathfrak{L}$ is periodic.

If $\mathfrak{N}$ is a nilpotent of class n subgroup or solvable of class s subgroup, or invariant subgroup, or non-invariant subgroup, either maximal abelian subgroup, then, as it was shown in [2], the image of $\mathfrak{N}$ under the homomorphism $\mathfrak{M} \rightarrow \mathfrak{M}/\mathfrak{L}$ will be the same. Further, if $\mathfrak{K}$ is an abelian subgroup of finite rank, containing $\mathfrak{L}$, then by Lemma 3 the quotient group $\mathfrak{K}/\mathfrak{L}$ has a finite rank. Hence, if the group $\mathfrak{M}$ satisfies one of the conditions 7) – 10), then the group $\mathfrak{M}/\mathfrak{L}$ satisfies the same condition. We have proved above that the group $\mathfrak{M}/\mathfrak{L}$ is periodic. Then, as shown while considering the case when CML Q is periodic, the group $\mathfrak{M}/\mathfrak{L}$ has a finite rank. The subgroup $\mathfrak{L}$ has a finite rank, then it follows from Lemma 2 that the group $\mathfrak{M}$ has a finite rank too. Consequently, the implications 7) $\rightarrow$ 6), 8) $\rightarrow$ 6), 9) $\rightarrow$ 6), 10) $\rightarrow$ 6) hold for CML Q. This completes the proof of Theorem 1.

We notice that in [16] is shown the equivalence of conditions 6), 7) of theorem for arbitrary periodic locally nilpotent group. The multiplication group $\mathfrak{M}$ of an arbitrary CML Q is locally nilpotent and if the CML Q is non-periodic, then the group $\mathfrak{M}$ is also non-periodic. However, unlike the multiplication group $\mathfrak{M}$, in [16] there is an example of non-periodic locally nilpotent group for which the conditions 6), 7) of the theorem are not equivalent.

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